TRANSFORMATIONS OF SURFACES Ω*

(SECOND MEMOIR)

BY

LUTHER PFAHLER EISENHART

In a former memoir† we developed a theory of transformations K of conjugate systems with equal point invariants into systems of the same kind. Subsequently‡ we considered the case in which the lines joining corresponding points on two surfaces in the relation of a transformation K form a normal congruence. These surfaces, being of a particular kind, are called *surfaces* C. We have shown that there are transformations of the K type, denoted by K_m, which transform a surface C into surfaces C. The surfaces $\bar{S}$, orthogonal to these normal congruences, are the surfaces Ω discussed by Demoulin.§ We showed that there exist transformations A_m of $\bar{S}$ into surfaces $\bar{S}_i$, such that $\bar{S}$ and a surface $\bar{S}_1$ envelope a two-parameter family of spheres, and in the correspondence between $\bar{S}$ and $\bar{S}_1$ thus established lines of curvature correspond. In the present memoir we extend our investigations concerning these transformations A_m, and more particularly apply the results to certain types of surfaces Ω.

Transformations A_m belong to the general class of transformations of Ribaucour, and in the first part of the paper we put the equations in a form possessed by all transformations of Ribaucour.

Guichard discovered|| a class of surfaces possessing the following characteristic property: If $\bar{S}$ is such a surface, there exists an associate surface $\bar{S}_1$ such the lines of curvature on the two surfaces have the same spherical representation, and the principal radii of curvature ρ_1, ρ_2 and ρ_1', ρ_2' of the respective surfaces are in the relation

$$\rho_1 \rho_2' + \rho_2 \rho_1' = \text{const.} \neq 0.$$

* Presented to the Society, December 28, 1915.

† These Transactions, vol. 15 (1914), pp. 397–430. This memoir will be referred to as M_1.

‡ These Transactions, vol. 16 (1915), pp. 275–310. This memoir will be referred to as M_2.

§ Comptes Rendus, vol. 153 (1911), pp. 590–593, 705–707.

|| Comptes Rendus, vol. 130 (1900), p. 159.

Demoulin* remarked that these surfaces are surfaces Ω. We apply the above general results to these *surfaces of Guichard* and show that they admit transformations A_m into surfaces of Guichard.†

The circles orthogonal to two surfaces in the relation of a transformation A_m form a cyclic system. We call the planes of these circles the circle-planes of the transformation. Beginning with § 8, we consider the case where for two or more transformations A_m the circle-planes are the same for these transformations. We find that there are *special surfaces* Ω depending upon six arbitrary constants which admit, in general, three transformations A_m such that the circle-planes are the same, and these *complementary* transformations are known when a special surface is given.

We have shown in M_2 that the transformations D_m of isothermic surfaces, discovered by Darboux‡ and studied at length by Bianchi,§ are transformations A_m. Applying the foregoing results to isothermic surfaces we are led to the special isothermic surfaces studied by Darboux and Bianchi in the memoirs indicated.

In like manner these results when applied to surfaces of Guichard lead to the special surfaces of Guichard previously studied by the author.||

In §§ 3 and 19 a study is made of the envelope of the circle-planes of complementary transformations. These surfaces are applicable to quadrics when $\bar{S}$ is a special isothermic surface or a special surface of Guichard.

In the closing section it is shown that for certain surfaces Ω the transformations A_m go in pairs, the circle-planes of any pair being the same. But the transformations cannot be found directly as in the case of the complementary transformations of special surfaces. The surfaces of Guichard are of this type.

1. Equations of a surface C

In accordance with the definition of a surface C, its cartesian coördinates, x, y, z, a complementary function t, and the function $x^2 + y^2 + z^2 - t^2$ are solutions of an equation of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial \theta}{\partial v} = 0. \tag{1}$$

* L. c., p. 707.

† These transformations are the same as those found by the author formerly by a method not based upon the underlying transformations K_m, Annali di matematica, ser. 3, vol. 22 (1914), pp. 191–248.

‡ Annales de l'École Normale Supérieure, ser. 3, vol. 16 (1899).

§ Annali di matematica, ser. 3, vol. 11 (1905), pp. 93–158.

|| Annali, l. c., pp. 235 et seq.

The fundamental functions E, F, G, D, D'' of C are given by*

$$E = \left(\frac{\partial t}{\partial u}\right)^2 + \frac{1}{\rho}, \qquad F = \frac{\partial t}{\partial u}\frac{\partial t}{\partial v}, \qquad G = \left(\frac{\partial t}{\partial v}\right)^2 + \frac{1}{\rho},$$

$$H^2 = EG - F^2 = \frac{1}{\rho}\left[\left(\frac{\partial t}{\partial u}\right)^2 + \left(\frac{\partial t}{\partial v}\right)^2\right] + \frac{1}{\rho^2}, \tag{2}$$

$$\frac{D}{H} = \frac{m_0(\lambda_0 - \rho\theta_0)}{T_0} - L, \qquad \frac{D''}{H} = \frac{m_0(\lambda_0 + \rho\theta_0)}{T_0} - M,$$

where

$$L = \frac{1}{H^2\rho}\left[\frac{\partial^2 t}{\partial u^2} + \frac{\partial t}{\partial u}\frac{\partial \log\sqrt{\rho}}{\partial u} - \frac{\partial t}{\partial v}\frac{\partial \log\sqrt{\rho}}{\partial v}\right],$$

$$M = \frac{1}{H^2\rho}\left[\frac{\partial^2 t}{\partial v^2} - \frac{\partial t}{\partial u}\frac{\partial \log\sqrt{\rho}}{\partial u} + \frac{\partial t}{\partial v}\frac{\partial \log\sqrt{\rho}}{\partial v}\right], \tag{3}$$

and

$$\frac{\partial T_0}{\partial u} = m_0(\lambda_0 - \rho\theta_0)\frac{\partial t}{\partial u}, \qquad \frac{\partial T_0}{\partial v} = m_0(\lambda_0 + \rho\theta_0)\frac{\partial t}{\partial v}, \tag{4}$$

θ_0 being the solution of (1) which determines a conjugate surface C_0 of C (cf. M_2, § 3), and λ_0 being determined by the equations

$$\frac{\partial\lambda_0}{\partial u} = -\rho\frac{\partial\theta_0}{\partial u}, \qquad \frac{\partial\lambda_0}{\partial v} = \rho\frac{\partial\theta_0}{\partial v}. \tag{5}$$

By means of the above equations and the expressions [41], it can be shown that the expressions (2) for D and D'' satisfy the Codazzi equations.† There remains the Gauss equation‡ to be satisfied, which is reducible in consequence of (2) to

$$\left[\frac{m_0^2(\lambda_0^2 - \rho^2\theta_0^2)}{T_0^2} - L\frac{m_0}{T_0}(\lambda_0 + \rho\theta_0) - M\frac{m_0}{T_0}(\lambda_0 - \rho\theta_0)\right]\left[\left(\frac{\partial t}{\partial u}\right)^2 + \left(\frac{\partial t}{\partial v}\right)^2 + \frac{1}{\rho}\right] = \frac{\partial^2 \log\sqrt{\rho}}{\partial u^2} + \frac{\partial^2 \log\sqrt{\rho}}{\partial v^2}. \tag{6}$$

Recapitulating these results, we have

THEOREM 1. *The analytical determination of surfaces C consists in finding five functions λ_0, θ_0, T_0, ρ, and t satisfying equations* (4), (5), (6) *and*

$$\frac{\partial^2 t}{\partial u \partial v} + \frac{\partial \log\sqrt{\rho}}{\partial v}\frac{\partial t}{\partial u} + \frac{\partial \log\sqrt{\rho}}{\partial u}\frac{\partial t}{\partial v} = 0. \tag{7}$$

---

* Cf. [35], [36], [43], [46]. A reference of this sort is to equations of the memoir M_2.

† E., p. 156. A reference of this sort is to the author's *Differential Geometry*, Ginn and Co., Boston, 1909.

‡ E., p. 155.

2. Equations of transformations A_m

The lines joining corresponding points on a surface C and the conjugate surface C_0 form a normal congruence. The orthogonal surfaces are surfaces Ω by definition. Let $\bar{S}$ be one of these surfaces. In M_2, § 8, we established a transformation of $\bar{S}$ of the Ribaucour type. In the present section we give another form to the equations of such transformations A_m.

If $\bar{X}, \bar{Y}, \bar{Z}$; $\bar{X}_1, \bar{Y}_1, \bar{Z}_1$; $\bar{X}_2, \bar{Y}_2, \bar{Z}_2$ denote the direction-cosines of the normal to $\bar{S}$ and of the tangents to its lines of curvature, we have from [9], [12], and [53]

$$\bar{X} = \frac{1}{T_0}(a_0 X_1 + b_0 X_2 + w_0 X) \tag{8}$$

and

$$\bar{X}_1 = \sqrt{\rho}\left(\frac{\partial x}{\partial u} + \frac{\partial t}{\partial u}\bar{X}\right), \qquad \bar{X}_2 = \sqrt{\rho}\left(\frac{\partial x}{\partial v} + \frac{\partial t}{\partial v}\bar{X}\right), \tag{9}$$

where X, Y, Z; X_1, Y_1, Z_1; X_2, Y_2, Z_2 are the direction-cosines of the normal to C and of the bisectors of the angles between the parametric curves on C.

In consequence of [7] and (8), equations (9) may be replaced by

$$\bar{X}_1 = \sqrt{\rho}\left[\left(\sqrt{E}\cos\omega + \frac{a_0}{T_0}\frac{\partial t}{\partial u}\right)X_1 + \left(-\sqrt{E}\sin\omega + \frac{b_0}{T_0}\frac{\partial t}{\partial u}\right)X_2 + \frac{w_0}{T_0}\frac{\partial t}{\partial u}X\right],$$

$$\bar{X}_2 = \sqrt{\rho}\left[\left(\sqrt{G}\cos\omega + \frac{a_0}{T_0}\frac{\partial t}{\partial v}\right)X_1 + \left(\sqrt{G}\sin\omega + \frac{b_0}{T_0}\frac{\partial t}{\partial v}\right)X_2 + \frac{w_0}{T_0}\frac{\partial t}{\partial v}X\right],$$

where 2ω is one of the angles between the parametric curves on C.

By means of the expressions [42] and [46] for a_0, b_0, w_0, namely

$$\begin{aligned} a_0 &= -\frac{T_0}{2\cos\omega}\left(\frac{1}{\sqrt{E}}\frac{\partial t}{\partial u} + \frac{1}{\sqrt{G}}\frac{\partial t}{\partial v}\right), \\ b_0 &= \frac{T_0}{2\sin\omega}\left(\frac{1}{\sqrt{E}}\frac{\partial t}{\partial u} - \frac{1}{\sqrt{G}}\frac{\partial t}{\partial v}\right), \qquad w_0 = \frac{T_0}{H\rho}, \end{aligned} \tag{10}$$

these are reducible to

$$\begin{aligned} \bar{X}_1 &= \frac{1}{\sin 2\omega\sqrt{E\rho}}\left(\sin\omega\, X_1 - \cos\omega\, X_2 + \frac{1}{\sqrt{G}}\frac{\partial t}{\partial u}X\right), \\ \bar{X}_2 &= \frac{1}{\sin 2\omega\sqrt{G\rho}}\left(\sin\omega\, X_1 + \cos\omega\, X_2 + \frac{1}{\sqrt{E}}\frac{\partial t}{\partial v}X\right). \end{aligned} \tag{11}$$

From (8) and (11) we have

$$
(12) \qquad
\begin{aligned}
X_1 &= \frac{a_0}{T_0}\bar{X} + \frac{1}{2\cos\omega}\left(\frac{1}{\sqrt{E\rho}}\bar{X}_1 + \frac{1}{\sqrt{G\rho}}\bar{X}_2\right),\\
X_2 &= \frac{b_0}{T_0}\bar{X} + \frac{1}{2\sin\omega}\left(-\frac{1}{\sqrt{E\rho}}\bar{X}_1 + \frac{1}{\sqrt{G\rho}}\bar{X}_2\right),\\
X &= \frac{w_0}{T_0}\bar{X} + \frac{1}{H\sqrt{\rho}}\left(\frac{\partial t}{\partial u}\bar{X}_1 + \frac{\partial t}{\partial v}\bar{X}_2\right).
\end{aligned}
$$

If $\bar{S}_1$ denotes a transform of the surface $\bar{S}$ by a transformation A_{m_1}, as developed in M_2, § 8, the two surfaces $\bar{S}$ and $\bar{S}_1$ are the envelope of a two-parameter family of spheres. From [98] and [99] it follows that the coördinates, ξ, η, ζ, of the centers of these spheres are given by equations of the form

$$(13) \qquad \xi = \bar{x} + (\sigma - t)\bar{X} = \bar{x}_1 + (\sigma - t)\bar{X}',$$

where $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$ are the cartesian coördinates of $\bar{S}_1$; $\bar{X}'$, $\bar{Y}'$, $\bar{Z}'$, are the direction cosines of the normal to $\bar{S}_1$; and

$$(14) \qquad \sigma = \frac{T_0\,\theta_1}{m_0\,\lambda_0(\theta_1 - \theta_{01})}.$$

In order to explain the terms in this expression, we remark that $\bar{S}_1$, being a surface Ω, is normal to the lines joining corresponding points on two surfaces C_1 and C_{10}. These surfaces C_1 and C_{10} are in the relations of transformations K_{m_1} with C and C_0 respectively, and the transformation functions are θ_1, λ_1 and θ_{01}, λ_{01} respectively.

If x_{10}, y_{10}, z_{10} and x_1, y_1, z_1 denote the cartesian coördinates of C_{10} and C_1 respectively, we have

$$(15) \qquad \bar{X}' = \frac{\lambda_{10}\,m_0}{T_{10}}(x_{10} - x_1), \qquad \bar{X} = \frac{\lambda_0\,m_0}{T_0}(x_0 - x),$$

where T_{10} is the analogue of T_0.

Since C, C_0, C_1, C_{10} form a quatern of surfaces under the relation of transformations K_m, we have from [73], [76], [77], [79] the relations

$$
(16) \qquad
\begin{gathered}
\lambda_{ij}\,\theta_i = \theta_{ij}\,\lambda_i - \theta_j\,\lambda_i + \theta_i\,\lambda_j, \qquad \theta_{ij}\,\lambda_i + \theta_{ji}\,\lambda_j = 0,\\
\lambda_{ij}\,\theta_i + \lambda_{ji}\,\theta_j = 0,
\end{gathered}
$$

and

$$(17) \qquad \lambda_{ij}\,\theta_i\,x_{ij} = \theta_{ij}\,\lambda_i\,x - \lambda_i\,\theta_j\,x_i + \lambda_j\,\theta_i\,x_j,$$

for $i = 1, j = 0$. Also from [9]

$$(18) \qquad x_i - x = \frac{1}{m_i\,\lambda_i}(a_i\,X_1 + b_i\,X_2 + w_i\,X)$$

for $i = 1$.

By means of these equations the expression (15) can be given the form

$$\overline{X}' = \frac{m_0}{\theta_1 T_{10}}[(\theta_{10}\lambda_1 + \lambda_0\theta_1)(x_1 - x) + \lambda_0\theta_1(x - x_0)]$$

$$= -\frac{m_0}{m_1 T_{10}}\left[\left(\frac{\theta_{01}-\theta_1}{\lambda_1 m_1}\lambda_0 a_1 + \frac{\theta_1 a_0}{m_0}\right)X_1 + \left(\frac{\theta_{01}-\theta_1}{\lambda_1 m_1}\lambda_0 b_1 + \frac{\theta_1 b_0}{m_0}\right)X_2 + \left(\frac{\theta_{01}-\theta_1}{\lambda_1 m_1}\lambda_0 w_1 + \frac{\theta_1 w_0}{m_0}\right)X\right]. \tag{19}$$

If the expressions (12) for X_1, X_2, and X be substituted in this equation and use be made of the identities

$$\begin{aligned} a_0 \sin\omega - b_0\cos\omega + \frac{w_0}{\sqrt{G}}\frac{\partial t}{\partial u} &= 0, \\ a_0 \sin\omega + b_0\cos\omega + \frac{w_0}{\sqrt{E}}\frac{\partial t}{\partial v} &= 0, \end{aligned} \tag{20}$$

which follow from [42], and of the relation [97], namely

$$(T_{10} - T_0)\theta_1 + (\theta_{01} - \theta_1)m_0\lambda_0(t_1 - t) = 0, \tag{21}$$

t_1 being the complementary function for C_1, we have

$$\begin{aligned} \overline{X}' = \overline{X} + \frac{m_0\lambda_0(\theta_1-\theta_{01})}{T_{10}\theta_1\lambda_1 m_1}\Bigg\{&\left[\frac{\Phi_{10}}{T_0} + \lambda_1 m_1(t_1 - t)\right]\overline{X} \\ &+ \frac{(a_1\sin\omega - b_1\cos\omega)\sqrt{G} + w_1\dfrac{\partial t}{\partial u}}{\sqrt{\rho H}}\overline{X}_1 \\ &+ \frac{(a_1\sin\omega + b_1\cos\omega)\sqrt{E} + w_1\dfrac{\partial t}{\partial v}}{\sqrt{\rho H}}\overline{X}_2\Bigg\}, \end{aligned} \tag{22}$$

where we use the abbreviation

$$\Phi_{ij} = a_i a_j + b_i b_j + w_i w_j \qquad (i \mp j). \tag{23}$$

From (13) we have accordingly

$$\bar{x}_1 - \bar{x} = (t - \sigma)(\overline{X}' - \overline{X}) = -\frac{1}{\sigma_1 m_1}(\eta_1\overline{X}_1 + \beta_1\overline{X}_2 + \mu_1\overline{X}), \tag{24}$$

where

$$\begin{aligned} \sigma_1 &= \frac{T_{10}\theta_1\lambda_1}{T_0\theta_1 + m_0\lambda_0(\theta_{01}-\theta_1)t}, \\ \eta_1 &= \frac{1}{\sqrt{\rho}H}\left[(a_1\sin\omega - b_1\cos\omega)\sqrt{G} + w_1\frac{\partial t}{\partial u}\right], \\ \beta_1 &= \frac{1}{\sqrt{\rho}H}\left[(a_1\sin\omega + b_1\cos\omega)\sqrt{E} + w_1\frac{\partial t}{\partial v}\right], \\ \mu_1 &= m_1\lambda_1(t_1 - t) + \frac{\Phi_{10}}{T_0}. \end{aligned} \tag{25}$$

From [100] it follows that the radius R of the variable sphere whose envelope is $\bar{S}$ and $\bar{S}_1$ is given by

$$R = \frac{m_0 \lambda_0 t (\theta_{01} - \theta_1) + T_0 \theta_1}{m_0 \lambda_0 (\theta_{01} - \theta_1)}. \tag{26}$$

We consider for a moment a general transformation of Ribaucour of any surface. If the equations of the transformation are written in the form (24), the functions $\eta_1, \beta_1, \mu_1, \sigma_1$ and the function τ_1 defined by

$$2m_1 \tau_1 \sigma_1 = \eta_1^2 + \beta_1^2 + \mu_1^2 \tag{27}$$

satisfy the equations:

$$\begin{aligned}
&\frac{\partial \mu_1}{\partial u} = -\sqrt{\bar{E}}\frac{\eta_1}{\rho_1}, && \frac{\partial \mu_1}{\partial v} = -\sqrt{\bar{G}}\frac{\beta_1}{\rho_2},\\
&\frac{\partial \tau_1}{\partial u} = \sqrt{\bar{E}}\,\eta_1, && \frac{\partial \tau_1}{\partial v} = \sqrt{\bar{G}}\,\beta_1,\\
&\frac{\partial \beta_1}{\partial u} = \frac{\eta_1}{\sqrt{\bar{G}}}\frac{\partial\sqrt{\bar{E}}}{\partial v}, && \frac{\partial \eta_1}{\partial v} = \frac{\beta_1}{\sqrt{\bar{E}}}\frac{\partial\sqrt{\bar{G}}}{\partial u},\\
&\frac{\partial \log \sigma_1}{\partial u} = -\sqrt{\bar{E}_1}\frac{\eta_1}{\tau_1}, && \frac{\partial \log \sigma_1}{\partial v} = \sqrt{\bar{G}_1}\frac{\beta_1}{\tau_1},\\
&\frac{\partial \eta_1}{\partial u} = -\frac{\beta_1}{\sqrt{\bar{G}}}\frac{\partial\sqrt{\bar{E}}}{\partial v} + \mu_1\frac{\sqrt{\bar{E}}}{\rho_1} + m_1\sigma_1(\sqrt{\bar{E}} - \sqrt{\bar{E}_1}),\\
&\frac{\partial \beta_1}{\partial v} = -\frac{\eta_1}{\sqrt{\bar{E}}}\frac{\partial\sqrt{\bar{G}}}{\partial u} + \mu_1\frac{\sqrt{\bar{G}}}{\rho_2} + m_1\sigma_1(\sqrt{\bar{G}} + \sqrt{\bar{G}_1}),
\end{aligned} \tag{28}$$

where $\bar{E}$, $\bar{G}$ and $\bar{E}_1$, $\bar{G}_1$ are the first fundamental coefficients of the given surface $\bar{S}$ and its transform $\bar{S}_1$, and ρ_1, ρ_2 are the principal radii of curvature of $\bar{S}$. Conversely, every set of functions satisfying these equations determines a transformation of Ribaucour.*

The ratio τ_1/μ_1 is equal to the radius of the variable sphere of the transformation. When $\bar{S}$ is a surface Ω and the transformation is of the type A_m, it follows from [93] that

$$m_0 \lambda_0 (\theta_{01} - \theta_1) + T_0 m_1 \lambda_1 (t_1 - t) + \Phi_{10} = 0. \tag{29}$$

Hence from (25), (26), and (29) we have

$$\begin{aligned}
\tau_1 &= -\frac{m_0 \lambda_0 (\theta_{01} - \theta_1) t + T_0 \theta_1}{T_0},\\
\mu_1 &= -\frac{m_0 \lambda_0 (\theta_{01} - \theta_1)}{T_0} = \frac{m_0 (\lambda_1 \theta_{10} + \lambda_0 \theta_1)}{T_0}.
\end{aligned} \tag{30}$$

* Annali, ser. 3, vol. 22 (1914), pp. 191–248.

From these expressions follows

$$\tau_1 = \mu_1 t - \theta_1, \tag{31}$$

and from (21) and (30)

$$\theta_1 T_{10} = T_0[\mu_1(t_1 - t) + \theta_1]. \tag{32}$$

Hence from the first of (25) we have

$$\sigma_1 = \frac{\lambda_1}{\tau_1}[\mu_1(t_1 - t) + \theta_1] = \frac{\lambda_1}{\tau_1}(\mu_1 t_1 - \tau_1). \tag{33}$$

From [54] we have that the fundamental functions for a surface $\bar{S}$ are expressible in the form

$$\begin{gathered}
\sqrt{\bar{E}} = \left[1 - \frac{m_0 t}{T_0}(\lambda_0 - \rho\theta_0)\right]\frac{1}{\sqrt{\rho}}, \\
\sqrt{\bar{G}} = \left[1 - \frac{m_0 t}{T_0}(\lambda_0 + \rho\theta_0)\right]\frac{1}{\sqrt{\rho}}, \\
\frac{\sqrt{\bar{E}}}{\rho_1} = \frac{m_0(\lambda_0 - \rho\theta_0)}{T_0\sqrt{\rho}}, \qquad \frac{\sqrt{\bar{G}}}{\rho_2} = \frac{m_0(\lambda_0 + \rho\theta_0)}{T_0\sqrt{\rho}}, \\
\bar{D} = \bar{E}/\rho_1, \qquad \bar{D}'' = \bar{G}/\rho_2.
\end{gathered} \tag{34}$$

When equation (31) is differentiated and use is made of equations (28) and (34), we obtain

$$\eta_1 = \sqrt{\rho}\left(\mu_1\frac{\partial t}{\partial u} - \frac{\partial\theta_1}{\partial u}\right), \qquad \beta_1 = \sqrt{\rho}\left(\mu_1\frac{\partial t}{\partial v} - \frac{\partial\theta_1}{\partial v}\right). \tag{35}$$

A function θ_1 determining a transformation K_{m_1} of C, and consequently a transformation A_{m_1} of $\bar{S}$, must satisfy the equations [65], namely

$$\begin{aligned}
&\frac{\partial\theta_1}{\partial u} + \sqrt{E}\left[\cos\omega\, a_1 - \sin\omega\, b_1 + (t_1 - t)\frac{m_1\lambda_1}{T_0}(\cos\omega\, a_0 \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. - \sin\omega\, b_0)\right] = 0, \\
&\frac{\partial\theta_1}{\partial v} + \sqrt{G}\left[\cos\omega\, a_1 + \sin\omega\, b_1 + (t_1 - t)\frac{m_1\lambda_1}{T_0}(\cos\omega\, a_0 \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + \sin\omega\, b_0)\right] = 0.
\end{aligned} \tag{36}$$

When these expressions are substituted in (35) and also the expression for μ_1 from (25), we obtain

$$\begin{aligned}
\eta_1 &= \frac{\sqrt{\rho}}{T_0}\left[\Phi_{10}\frac{\partial t}{\partial u} + T_0\sqrt{E}(a_1\cos\omega - b_1\sin\omega)\right], \\
\beta_1 &= \frac{\sqrt{\rho}}{T_0}\left[\Phi_{10}\frac{\partial t}{\partial v} + T_0\sqrt{G}(a_1\cos\omega + b_1\sin\omega)\right].
\end{aligned} \tag{37}$$

These equations are consistent with (25) in consequence of (10).

We recall from M_2, § 6, that the complementary function t_1 of C_1 is given by

$$(38)\qquad \begin{aligned} \frac{\partial t_i}{\partial u} &= \frac{\rho}{\lambda_i}\left[(t_i - t)\frac{\partial \theta_i}{\partial u} + \theta_i \frac{\partial t}{\partial u}\right], \\ \frac{\partial t_i}{\partial v} &= -\frac{\rho}{\lambda_i}\left[(t_i - t)\frac{\partial \theta_i}{\partial v} + \theta_i \frac{\partial t}{\partial v}\right], \end{aligned}$$

for $i = 1$. Making use of these equations and the preceding equations, we obtain from (33) by differentiation

$$(39)\qquad \begin{aligned} \frac{\partial \sigma_1}{\partial u} &= \frac{\eta_1}{\sqrt{\rho}\tau_1^2}\left[\frac{\lambda_1 \theta_1 t_1 m_0 (\lambda_0 - \rho\theta_0)}{T_0} - \lambda_1 \mu_1 t_1 + \rho\theta_1 \tau_1 \right], \\ \frac{\partial \sigma_1}{\partial v} &= \frac{\beta_1}{\sqrt{\rho}\tau_1^2}\left[\frac{\lambda_1 \theta_1 t_1 m_0 (\lambda_0 + \rho\theta_0)}{T_0} - \lambda_1 \mu_1 t_1 - \rho\theta_1 \tau_1 \right]. \end{aligned}$$

From (28), (35), and (39) we have

$$(40)\qquad \begin{aligned} \sqrt{E_1} &= \frac{1}{\sqrt{\rho}(\tau_1 - \mu_1 t_1)}\left[\frac{\theta_1 t_1 m_0 (\lambda_0 - \rho\theta_0)}{T_0} - t_1\mu_1 + \frac{\rho\theta_1 \tau_1}{\lambda_1} \right], \\ \sqrt{G_1} &= \frac{-1}{\sqrt{\rho}(\tau_1 - \mu_1 t_1)}\left[\frac{\theta_1 t_1 m_0 (\lambda_0 + \rho\theta_0)}{T_0} - t_1\mu_1 - \frac{\rho\theta_1 \tau_1}{\lambda_1} \right]. \end{aligned}$$

From these expressions and (34) we get

$$(41)\qquad \begin{aligned} \sqrt{E_1} - \sqrt{E} &= \frac{\tau_1}{\lambda_1\sqrt{\rho}(\tau_1 - \mu_1 t_1)}\left[\lambda_1 (t - t_1)\frac{m_0(\lambda_0 - \rho\theta_0)}{T_0} - \lambda_1 + \rho\theta_1 \right], \\ \sqrt{G_1} + \sqrt{G} &= \frac{-\tau_1}{\lambda_1\sqrt{\rho}(\tau_1 - \mu_1 t_1)}\left[\lambda_1 (t - t_1)\frac{m_0(\lambda_0 + \rho\theta_0)}{T_0} - \lambda_1 - \rho\theta_1 \right]. \end{aligned}$$

Hence the last two of equations (28) may be written

$$(42)\qquad \begin{aligned} \frac{\partial \eta_1}{\partial u} &= -\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\beta_1 - \frac{m_1}{\sqrt{\rho}}(\lambda_1 - \rho\theta_1) + \Phi_{10}\frac{m_0(\lambda_0 - \rho\theta_0)}{\sqrt{\rho}T_0^2}, \\ \frac{\partial \beta_1}{\partial v} &= -\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\eta_1 - \frac{m_1}{\sqrt{\rho}}(\lambda_1 + \rho\theta_1) + \Phi_{10}\frac{m_0(\lambda_0 + \rho\theta_0)}{\sqrt{\rho}T_0^2}. \end{aligned}$$

To these formulas we add

$$(42')\qquad \frac{\Phi_{10}}{T_0} + \sqrt{\rho}\left(\eta_1 \frac{\partial t}{\partial u} + \beta_1 \frac{\partial t}{\partial v}\right) = H\rho w_1,$$

which is obtained by solving the second and third of equations (25) for a_1 and b_1 and by substituting these and the expressions (10) for a_0, b_0, w_0 in (23) for $i = 1, j = 0$.

3. Envelope of the circle-plane of a transformation A_m

It is a well-known fact that the circles orthogonal to two surfaces which are in the relation of a transformation of Ribaucour form a cyclic system, that is they are normal to an infinity of surfaces. We call the plane of this circle the *circle-plane of the transformation* and in this section we derive certain results concerning the surface S_0 which is the envelope of these planes.

From (24) it follows that the direction cosines of the normal to this plane are proportional to

$$\beta_1 \bar{X}_1 - \eta_1 \bar{X}_2, \qquad \beta_1 \bar{Y}_1 - \eta_1 \bar{Y}_2, \qquad \beta_1 \bar{Z}_1 - \eta_1 \bar{Z}_2. \tag{43}$$

Consequently the coördinates x_0, y_0, z_0, of S_0 are given by

$$x_0 = \bar{x} + p(\eta_1 \bar{X}_1 + \beta_1 \bar{X}_2) + q\bar{X}, \tag{44}$$

where p and q are to be determined. The conditions necessary and sufficient that S_0 be the envelope are that

$$\sum (\beta_1 \bar{X}_1 - \eta_1 \bar{X}_2) \frac{\partial x_0}{\partial u} = 0, \qquad \sum (\beta_1 \bar{X}_1 - \eta_1 \bar{X}_2) \frac{\partial x_0}{\partial v} = 0. \tag{45}$$

The derivatives of $\bar{X}_1, \bar{X}_2, \bar{X}$ are given by

$$\begin{aligned}
&\frac{\partial \bar{X}_1}{\partial u} = -\frac{1}{\sqrt{\bar{G}}} \frac{\partial \sqrt{\bar{E}}}{\partial v} \bar{X}_2 + \frac{\sqrt{\bar{E}}}{\rho_1} \bar{X}, \qquad && \frac{\partial \bar{X}_1}{\partial v} = \frac{1}{\sqrt{\bar{E}}} \frac{\partial \sqrt{\bar{G}}}{\partial u} \bar{X}_2, \\
&\frac{\partial \bar{X}_2}{\partial u} = \frac{1}{\sqrt{\bar{G}}} \frac{\partial \sqrt{\bar{E}}}{\partial v} \bar{X}_1, && \frac{\partial \bar{X}_2}{\partial v} = -\frac{1}{\sqrt{\bar{E}}} \frac{\partial \sqrt{\bar{G}}}{\partial u} \bar{X}_1 + \frac{\sqrt{\bar{G}}}{\rho_2} \bar{X}, \\
&\frac{\partial \bar{X}}{\partial u} = -\frac{\sqrt{\bar{E}}}{\rho_1} \bar{X}_1, && \frac{\partial \bar{X}}{\partial v} = -\frac{\sqrt{\bar{G}}}{\rho_2} \bar{X}_2.*
\end{aligned} \tag{46}$$

By means of these equations and (28) we find that equations (45) are equivalent to

$$\begin{aligned}
p &= \frac{1}{\bar{r}} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right), \\
q &= \frac{1}{\bar{r}} \left[\mu_1 \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) + m_1 \sigma_1 \left(\frac{\sqrt{\bar{G}_1}}{\sqrt{\bar{G}}} + \frac{\sqrt{\bar{E}_1}}{\sqrt{\bar{E}}} \right) \right], \\
\bar{r} &= m_1 \sigma_1 \left[\frac{1}{\rho_1} \left(1 + \frac{\sqrt{\bar{G}_1}}{\sqrt{\bar{G}}} \right) - \frac{1}{\rho_2} \left(1 - \frac{\sqrt{\bar{E}_1}}{\sqrt{\bar{E}}} \right) \right].
\end{aligned} \tag{47}$$

* E., p. 157.

From (28) we have

$$\frac{\partial \tau_1}{\partial u} + \rho_1 \frac{\partial \mu_1}{\partial u} = 0, \qquad \frac{\partial \tau_1}{\partial v} + \rho_2 \frac{\partial \mu_1}{\partial v} = 0. \tag{48}$$

Since these equations become the Rodrigues equations,* when τ_1 and μ_1 are replaced by $\bar{x}, \bar{y}, \bar{z}$ and $\overline{X}, \overline{Y}, \overline{Z}$ respectively, it follows that τ_1 is a solution of the point equation of $\overline{S}$ and μ_1 of the tangential equation. Hence $\mu_1, \overline{X}, \overline{Y}, \overline{Z}$ are the tangential coördinates of a surface Σ_0 which corresponds with parallelism of tangent planes to $\overline{S}$ and with lines of curvature in correspondence. If ξ_0, η_0, ζ_0 are the cartesian coördinates of Σ_0, their expressions as given by the general equations for tangential coördinates† are reducible in consequence of (28) and (46) to the form

$$\xi_0 = \mu_1 \overline{X} + \eta_1 \overline{X}_1 + \beta_1 \overline{X}_2. \tag{49}$$

Making use of these results, we have from (44)

$$dx_0 = \overline{X} d\omega + \xi_0 \, dp, \tag{50}$$

where

$$\omega = q - p\mu_1 = \frac{m_1 \sigma_1}{r} \left(\frac{\sqrt{G_1}}{\sqrt{G}} + \frac{\sqrt{E_1}}{\sqrt{E}} \right). \tag{50$'$}$$

If we put

$$\psi = \mu_1 \omega + \frac{p}{2}(\mu_1^2 + \eta_1^2 + \beta_1^2) + \tau_1, \tag{51}$$

the linear element of S_0 may be given the form

$$ds_0^2 = d\omega^2 + 2dpd\psi. \tag{52}$$

These results which are true for any cyclic system will be applied in § 19 to certain transformations A_m.

4. Inverse of a transformation A_m

It is evident that the relation of a transformation A_m is entirely reciprocal, and in the subsequent theory it will be desirable for us to know the functions by means of which $\overline{S}$ is given as a transform of $\overline{S}_1$.

The equation analogous to (24) is

$$\bar{x} - \bar{x}_1 = -\frac{1}{\sigma_1^{-1} m_1} (\eta_1^{-1} \overline{X}_1' + \beta_1^{-1} \overline{X}_2' + \mu_1^{-1} \overline{X}'),$$

where $\overline{X}', \overline{Y}', \overline{Z}'; \overline{X}_1', \overline{Y}_1', \overline{Z}_1'; \overline{X}_2', \overline{Y}_2', \overline{Z}_2'$ are the direction-cosines of the normal to $\overline{S}_1$ and of the tangents to its lines of curvature. For any trans-

* E., p. 122.

† E., p. 163

formation of Ribaucour, defined by (24), these direction cosines are given by the following expressions:*

$$\bar{X}_1' = \bar{X}_1 + \frac{\eta_1}{\tau_1}(\bar{x}_1 - \bar{x}), \qquad \bar{X}_2' = -\bar{X}_2 - \frac{\beta_1}{\tau_1}(\bar{x}_1 - \bar{x}),$$

$$\bar{X}' = \bar{X} + \frac{\mu_1}{\tau_1}(\bar{x}_1 - \bar{x}).$$

Substituting these expressions in the above equation and replacing $\bar{x}_1 - \bar{x}$ by its value from (24), we get an equation of the form

$$A\bar{X} + B\bar{X}_1 + C\bar{X}_2 = 0.$$

Since this equation is also true when the X's are replaced by the Y's and Z's, it follows that A, B, and C must be zero. This gives the three equations

$$\sigma_1 \frac{\mu_1^{-1}}{\mu_1} + \sigma_1^{-1} = \sigma_1 \frac{\eta_1^{-1}}{\eta_1} + \sigma_1^{-1} = -\sigma_1 \frac{\beta_1^{-1}}{\beta_1} + \sigma_1^{-1} = \frac{\mu_1 \mu_1^{-1} + \eta_1 \eta_1^{-1} - \beta_1 \beta_1^{-1}}{m_1 \tau_1}.$$

From these equations and (27) follow

$$\eta_1^{-1} = \rho\eta_1, \qquad \beta_1^{-1} = -\rho\beta_1, \qquad \mu_1^{-1} = \rho\mu_1, \qquad \sigma_1^{-1} = \rho\sigma_1,$$

where ρ is a factor of proportionality to be determined.

We have remarked that the radius of the variable sphere is τ_1/μ_1. Hence $\tau_1^{-1} = \rho\tau_1$. If this value be substituted in equations analogous to the third and fourth of (28), we have in consequence of others of these equations that $\rho = c/\tau_1\sigma_1$, where c denotes a constant. Hence

$$\begin{aligned} \eta_1^{-1} &= c\eta_1/\tau_1\sigma_1, \qquad \beta_1^{-1} = -c\beta_1/\tau_1\sigma_1, \qquad \mu_1^{-1} = c\mu_1/\tau_1\sigma_1, \\ \tau_1^{-1} &= c/\sigma_1, \qquad \sigma_1^{-1} = c/\tau_1. \end{aligned} \tag{53}$$

It is readily found that these values satisfy equations analogous to (28).

5. Surfaces of Guichard

It is our purpose to apply the preceding results to the surfaces of Guichard, as defined in the introduction. Calapso† made a study of these surfaces and determined their characterization. He found that there are two types which he called *surfaces of Guichard of the first and second kinds.* He showed that the fundamental functions of a surface of the first kind are given by

$$\begin{aligned} (54)\qquad \sqrt{\bar{E}} &= e^\xi \sinh\alpha, \qquad \sqrt{\bar{G}} = e^\xi \cosh\alpha, \qquad \bar{F} = 0, \qquad \bar{D}' = 0, \\ \bar{D} &= e^\xi \sinh\alpha(\cosh\alpha + h\sinh\alpha), \\ \bar{D}'' &= e^\xi \cosh\alpha(\sinh\alpha + h\cosh\alpha), \end{aligned}$$

* Annali, l. c., p. 196.

† Annali di matematica, ser. 3, vol. 11 (1905), pp. 201 et seq.

where ξ, α, and h are functions satisfying

$$\frac{\partial h}{\partial u} = (h + \coth \alpha) \frac{\partial \xi}{\partial u}, \qquad \frac{\partial h}{\partial v} = (h + \tanh \alpha) \frac{\partial \xi}{\partial v},$$

$$(55) \quad \begin{aligned} &\frac{\partial^2 \alpha}{\partial u^2} + \frac{\partial^2 \alpha}{\partial v^2} + \coth \alpha \frac{\partial^2 \xi}{\partial u^2} + \tanh \alpha \frac{\partial^2 \xi}{\partial v^2} - \operatorname{csch}^2 \alpha \frac{\partial \alpha}{\partial u} \frac{\partial \xi}{\partial u} \\ &\qquad + \operatorname{sech}^2 \alpha \frac{\partial \alpha}{\partial v} \frac{\partial \xi}{\partial v} + (\cosh \alpha + h \sinh \alpha)(\sinh \alpha + h \cosh \alpha) = 0, \end{aligned}$$

$$\frac{\partial^2 \xi}{\partial u \partial v} = \frac{\partial \xi}{\partial u} \frac{\partial \xi}{\partial v} + \coth \alpha \frac{\partial \xi}{\partial u} \frac{\partial \alpha}{\partial v} + \tanh \alpha \frac{\partial \xi}{\partial v} \frac{\partial \alpha}{\partial u}.$$

Conversely every set of functions satisfying these equations determines such a surface. In what follows we consider only surfaces of the first kind, but analogous results are equally true for surfaces of the second kind.

The fundamental functions of the associate surface $\bar{S}'$ are of the same form as (54) in functions ξ', α', h', given by

$$(56) \quad \begin{aligned} e^{\xi'} &= e^{-\xi}(1 - h^2), \\ \sinh \alpha' &= \frac{1}{h^2 - 1}[\sinh \alpha (1 + h^2) + 2h \cosh \alpha], \\ \cosh \alpha' &= \frac{1}{1 - h^2}[\cosh \alpha (1 + h^2) + 2h \sinh \alpha]. \end{aligned}$$

Comparing equations (34) and (54), we have

$$(57) \quad \begin{aligned} m_0 (\lambda_0 - \rho\theta_0) &= \sqrt{\rho} T_0 (\cosh \alpha + h \sinh \alpha), \\ m_0 (\lambda_0 + \rho\theta_0) &= \sqrt{\rho} T_0 (\sinh \alpha + h \cosh \alpha), \\ T_0 - m_0 t (\lambda_0 - \rho\theta_0) &= T_0 \sqrt{\rho} e^{\xi} \sinh \alpha, \\ T_0 - m_0 t (\lambda_0 + \rho\theta_0) &= T_0 \sqrt{\rho} e^{\xi} \cosh \alpha. \end{aligned}$$

Eliminating $\lambda_0 - \rho\theta_0$ and $\lambda_0 + \rho\theta_0$ from these equations, we obtain two equations which are equivalent to

$$(58) \qquad t = e^{\xi}/(1 - h), \qquad \sqrt{\rho} = (1 - h) e^{-(\xi + \alpha)}.$$

Substituting these values in (57), we find

$$(59) \qquad \lambda_0 = \frac{1}{2} \frac{T_0}{m_0} (1 - h^2) e^{-\xi}, \qquad \theta_0 = -\frac{1}{2} \frac{T_0}{m_0} e^{\xi}.$$

With the aid of the first two of equations (57) the first two of equations (4)

are reducible to

$$
(60)\qquad \begin{aligned}
\frac{\partial \log T_0}{\partial u} &= \sqrt{\rho}\,(\cosh \alpha + h \sinh \alpha)\,\frac{\partial t}{\partial u},\\
\frac{\partial \log T_0}{\partial v} &= \sqrt{\rho}\,(\sinh \alpha + h \cosh \alpha)\,\frac{\partial t}{\partial v}.
\end{aligned}
$$

From (58) and (55) we have

$$
(61)\qquad \frac{\partial t}{\partial u} = e^{\xi+\alpha}\frac{\operatorname{csch} \alpha}{(1-h)^2}\frac{\partial \xi}{\partial u}, \qquad \frac{\partial t}{\partial v} = e^{\xi+\alpha}\frac{\operatorname{sech} \alpha}{(1-h)^2}\frac{\partial \xi}{\partial v}.
$$

Substituting these values in (60) and making use of (55), we find

$$
(62)\qquad \frac{\partial \log T_0}{\partial u} = \frac{1}{1-h}\frac{\partial h}{\partial u}, \qquad \frac{\partial \log T_0}{\partial v} = \frac{1}{1-h}\frac{\partial h}{\partial v}.
$$

Since λ_0, θ_0, T_0 in the transformation from C to C_0 are determined only to within a constant factor, it follows that in all generality equations (62) may be replaced by

$$
(63)\qquad T_0 = \frac{1}{1-h}.
$$

Consequently equations (59) become

$$
(64)\qquad \lambda_0 = \frac{1}{2}\frac{1+h}{m_0}e^{-\xi}, \qquad \theta_0 = \frac{1}{2}\frac{e^{\xi}}{(h-1)m_0}.
$$

It is readily shown that these functions satisfy equations (5).

From (58) and (55) follow also

$$
(65)\qquad \begin{aligned}
\frac{\partial \log \sqrt{\rho}}{\partial u} &= \frac{e^{\alpha}}{h-1}\operatorname{csch} \alpha\,\frac{\partial \xi}{\partial u} - \frac{\partial \alpha}{\partial u},\\
\frac{\partial \log \sqrt{\rho}}{\partial v} &= \frac{e^{\alpha}}{h-1}\operatorname{sech} \alpha\,\frac{\partial \xi}{\partial v} - \frac{\partial \alpha}{\partial v}.
\end{aligned}
$$

With the aid of these results it can be shown that t, given by (58), is a solution of equation (1).

From the general existence theorem concerning surfaces of Guichard, as given by Calapso, it follows that each set of functions ξ, α, h, satisfying equations (55), determine a surface C, for which t and ρ are given by (58), and the surface C_0 is determined by the functions λ_0, θ_0, T_0, given by (63) and (64).

6. Transformation of surfaces of Guichard

It is our purpose now to show that there are transformations A_m of surfaces of Guichard into surfaces of the same kind, and that these transformations

are equivalent to certain ones obtained by us* previously in another manner.

In the first place we observe that there must exist two functions λ_{10}, θ_{10} for the new surface whose expressions are of the form (64), namely

$$\lambda_{10} = \frac{1}{2}\frac{1+h_1}{m_0}e^{-\xi_1}, \qquad \theta_{10} = \frac{1}{2}\frac{e^{\xi_1}}{(h_1-1)m_0}. \tag{66}$$

When the values (64) and (66) are substituted in the first of equations (16) for $i = 1, j = 0$, we have

$$\left(e^{-\xi_1}(1+h_1) - (1+h)e^{-\xi}\right)\theta_1 = \lambda_1\left(\frac{e^{\xi_1}}{h_1-1} - \frac{e^{\xi}}{h-1}\right). \tag{67}$$

From the general theory of transformations K† it follows that the point equation for C_1 is that obtained from (1) when ρ is replaced by ρ_1, where

$$\rho_1 = \frac{\lambda_1^2}{\rho\theta_1^2}. \tag{68}$$

From (58) it follows that we must have for the surface C_1

$$\sqrt{\rho_1} = (1-h_1)e^{-(\xi_1+\alpha_1)}. \tag{69}$$

When this value and that for ρ from (58) are substituted in (68), we get

$$\lambda_1 = \theta_1(1-h_1)(1-h)e^{-(\xi_1+\alpha_1+\xi+\alpha)}. \tag{70}$$

Eliminating λ_1 from equations (67) and (70), we get

$$(1-e^{-(\alpha_1+\alpha)})(e^{\xi}-e^{\xi_1}) + (1+e^{-(\alpha_1+\alpha)})(h_1e^{\xi} - he^{\xi_1}) = 0.$$

If we introduce a function k_1 by the equation

$$e^{\alpha_1+\alpha} = \frac{1-k_1}{1+k_1}, \tag{71}$$

the above equation may be replaced by

$$h = k_1 - l_1e^{\xi}, \qquad h_1 = k_1 - l_1e^{\xi_1}, \tag{72}$$

l_1 being a function thus defined.

From (71) it follows that

$$\begin{aligned} \cosh\alpha_1(1-k_1^2) &= (1+k_1^2)\cosh\alpha + 2k_1\sinh\alpha, \\ \sinh\alpha_1(1-k_1^2) &= -2k_1\cosh\alpha - (1+k_1^2)\sinh\alpha, \end{aligned} \tag{73}$$

and thence

$$\begin{aligned} \cosh\alpha_1 + k_1\sinh\alpha_1 &= \cosh\alpha + k_1\sinh\alpha = \phi_1, \\ \sinh\alpha_1 + k_1\cosh\alpha_1 &= -(\sinh\alpha + k_1\cosh\alpha) = -\psi_1, \end{aligned} \tag{74}$$

* L. c.

† M_1, p. 412.

ϕ_1 and ψ_1 being thus defined. Hence (73) may be written

$$(75)\qquad \begin{aligned}(1-k_1^2)\cosh\alpha_1 &= \phi_1 + k_1\psi_1,\\ (1-k_1^2)\sinh\alpha_1 &= -(\psi_1 + k_1\phi_1).\end{aligned}$$

From (54) and (58) it follows that t_1 and the fundamental quantities $\overline{E}_1$ and $\overline{G}_1$ for $\overline{S}_1$ must be of the form

$$(76)\qquad \sqrt{\overline{E}_1} = e^{\xi_1}\sinh\alpha_1, \qquad \sqrt{\overline{G}_1} = e^{\xi_1}\cosh\alpha_1, \qquad t_1 = \frac{e^{\xi_1}}{1-h_1}.$$

When these values are substituted in (40), the resulting equations are reducible by means of (57), (58), (72), and (74) to the single equation

$$(77)\qquad l_1\theta_1(1-h) + \mu_1(1-k_1) = 0.$$

From this and (31) we obtain

$$(78)\qquad \theta_1(1-h) + \tau_1(1-k_1) = 0,$$

and consequently

$$(79)\qquad \mu_1 = l_1\tau_1,$$

that is, $1/l_1$ is the radius of the variable sphere.

When the above values of λ_1, τ_1, μ_1, and t_1 are substituted in (33), we get

$$(80)\qquad \sigma_1 = \tau_1 e^{-(\xi_1+\xi)}(1-k_1^2).$$

Hence for the type of transformations A_m now under discussion equations (28) are

$$(81)\qquad \begin{aligned}
&\frac{\partial\sigma_1}{\partial u} = e^{-\xi}\eta_1(k_1\phi_1+\psi_1), \qquad \frac{\partial\sigma_1}{\partial v} = e^{-\xi}\beta_1(k_1\psi_1+\phi_1),\\
&\frac{\partial\tau_1}{\partial u} = e^{\xi}\eta_1\sinh\alpha, \qquad \frac{\partial\tau_1}{\partial v} = e^{\xi}\beta_1\cosh\alpha,\\
&\frac{\partial\mu_1}{\partial u} = -\eta_1(\cosh\alpha + h\sinh\alpha), \qquad \frac{\partial\mu_1}{\partial v} = -\beta_1(\sinh\alpha + h\cosh\alpha),\\
&\frac{\partial\eta_1}{\partial u} = -\beta_1\left(\tanh\alpha\frac{\partial\xi}{\partial v} + \frac{\partial\alpha}{\partial v}\right) + \mu_1(\cosh\alpha + h\sinh\alpha)\\
&\qquad\qquad + m_1\sigma_1 e^{\xi}\sinh\alpha + m_1\tau_1 e^{-\xi}(k_1\phi_1+\psi_1),\\
&\frac{\partial\eta_1}{\partial v} = \beta_1\left(\coth\alpha\frac{\partial\xi}{\partial u} + \frac{\partial\alpha}{\partial u}\right), \qquad \frac{\partial\beta_1}{\partial u} = \eta_1\left(\tanh\alpha\frac{\partial\xi}{\partial v} + \frac{\partial\alpha}{\partial v}\right),\\
&\frac{\partial\beta_1}{\partial v} = -\eta_1\left(\coth\alpha\frac{\partial\xi}{\partial u} + \frac{\partial\alpha}{\partial u}\right) + \mu_1(\sinh\alpha + h\cosh\alpha)\\
&\qquad\qquad + m_1\sigma_1 e^{\xi}\cosh\alpha + m_1\tau_1 e^{-\xi}(k_1\psi_1+\phi_1).
\end{aligned}$$

In consequence of (79) equations (72) are equivalent to

$$h = k_1 - e^{\xi}\frac{\mu_1}{\tau_1}, \qquad h_1 = k_1 - e^{\xi_1}\frac{\mu_1}{\tau_1}. \tag{82}$$

When these values of h and h_1 respectively are substituted in the first two of equations (55) and analogous equations in functions with the subscript 1, we get

$$\begin{aligned} \frac{\partial k_1}{\partial u} &= \phi_1\left(\operatorname{csch}\alpha\frac{\partial\xi}{\partial u} + e^{\xi}\frac{\eta_1}{\tau_1}\right), \\ \frac{\partial k_1}{\partial v} &= \psi_1\left(\operatorname{sech}\alpha\frac{\partial\xi}{\partial v} + e^{\xi}\frac{\beta_1}{\tau_1}\right), \end{aligned} \tag{83}$$

and

$$\begin{aligned} \frac{\partial \xi_1}{\partial u} &= \sinh\alpha_1\left(\operatorname{csch}\alpha\frac{\partial\xi}{\partial u} + (e^{\xi_1} - e^{\xi})\frac{\eta_1}{\tau_1}\right), \\ \frac{\partial \xi_1}{\partial v} &= -\cosh\alpha_1\left(\operatorname{sech}\alpha\frac{\partial\xi}{\partial v} + (e^{\xi_1} - e^{\xi})\frac{\beta_1}{\tau_1}\right). \end{aligned} \tag{84}$$

When the value of e^{ξ_1} from equation (80) is substituted in these equations, they are satisfied identically in consequence of the above equations.

When the expression for k_1 from (71) is substituted in (83), we get

$$\begin{aligned} \frac{\partial}{\partial u}(\alpha_1 + \alpha) &= -(\cosh\alpha_1 + \cosh\alpha)\left(\operatorname{csch}\alpha\frac{\partial\xi}{\partial u} - e^{\xi}\frac{\eta_1}{\tau_1}\right), \\ \frac{\partial}{\partial v}(\alpha_1 + \alpha) &= (\sinh\alpha_1 - \sinh\alpha)\left(\operatorname{sech}\alpha\frac{\partial\xi}{\partial v} - e^{\xi}\frac{\beta_1}{\tau_1}\right). \end{aligned}$$

It is readily shown that these values of α_1 and ξ_1 satisfy also equations analogous to the last two of (55). Hence we have

THEOREM 2. *If $\bar{S}$ is a surface of Guichard of the first kind, each set of functions $\sigma_1, \tau_1, \mu_1, \eta_1, \beta_1$ satisfying* (81) *and* (27), *where k_1, ϕ_1, and ψ_1 are given by* (74) *and* (82), *determines a transformation A_{m_1} from $\bar{S}$ into a surface of the same kind; the functions α_1, ξ_1, h_1, of $\bar{S}_1$ are given by* (73), (80), *and* (82). Hence in addition to the constants m_1 there are three arbitrary constants of integration.

From (31), (58), (70), (71), (80), and (82), we get

$$\begin{aligned} \theta_1 &= \frac{\tau_1}{1-h}(k_1 - 1), \\ \lambda_1 &= -\frac{\sigma_1(1-h_1)}{1-k_1} = -\sigma_1 - \mu_1 e^{-\xi}(1+k_1), \end{aligned} \tag{85}$$

as the functions determining the transformation of the surface C associated with $\bar{S}$.

7. The associate surface of Guichard

In M_2, § 10, we found that the transformation A_{m_1}, for which

$$(86)\quad \theta_1 = -\lambda_1 = 1, \qquad \theta_{10} = \lambda_0, \qquad \lambda_{10} = \theta_0, \qquad \theta_{01} = -\lambda_{01} = 1,$$

transforms a surface $\bar{S}$ into a surface $\bar{S}_1$ corresponding to $\bar{S}$ with parallelism of tangent planes, instead of the surfaces $\bar{S}$ and $\bar{S}_1$ being the envelope of a two-parameter family of spheres. We shall show that if $\bar{S}$ is a surface of Guichard the surface $\bar{S}_1$ so obtained is the associate surface referred to in the definition of surfaces of Guichard, as given in the introduction.

From [111] we have

$$(87)\qquad T_{10} = T_0.$$

On the assumption that $\bar{S}_1$ is a surface of Guichard we have from (63) and a similar expression for T_{10}

$$(88)\qquad h_1 = h.$$

Likewise from (64), (66), and (86) we have

$$(89)\qquad e^{\xi_1} = (1 - h^2)\, e^{-\xi}.$$

From the first two of equations (55) and similar equations for $\bar{S}_1$ we have in consequence of (88)

$$(h + \coth \alpha_1) \frac{\partial \xi_1}{\partial u} = (h + \coth \alpha) \frac{\partial \xi}{\partial u},$$

$$(h + \tanh \alpha_1) \frac{\partial \xi_1}{\partial v} = (h + \tanh \alpha) \frac{\partial \xi}{\partial v}.$$

If the expression for e^{ξ_1} from (89) is substituted in these equations, it is found that they are equivalent to

$$(90)\qquad \begin{aligned} \sinh \alpha_1 &= \frac{1}{h^2 - 1} [\sinh \alpha (1 + h^2) + 2h \cosh \alpha], \\ \cosh \alpha_1 &= \frac{-1}{h^2 - 1} [\cosh \alpha (1 + h^2) + 2h \sinh \alpha]. \end{aligned}$$

But these equations and (88) define the associate surface of $\bar{S}$, as Calapso has shown.* Hence our assumption has been justified and we have

Theorem 3. *A surface of Guichard and its associate are in the relation of the parallel transformation A_m, that is, the one for which $\theta_1 = -\lambda_1 = 1$.*

8. Transformations A_m with the same circle-planes

We begin the study now of surfaces Ω which admit several transformations A_m with the same circle-planes.

* L. c., p. 214.

From the expressions (43) of the direction-parameters of the circle-plane of a transformation A_m it follows that if a second transformation A_m of $\bar{S}$ has the same circle-planes as the original transformation, the corresponding functions β_2 and η_2 are proportional to β_1 and η_1 respectively. From equations (28) and similar ones for the second transformation of $\bar{S}$, we note that we must have

$$\frac{\partial(\tau_1, \tau_2)}{\partial(u, v)} = 0, \qquad \frac{\partial(\mu_1, \mu_2)}{\partial(u, v)} = 0.$$

Consequently τ_2 is a function of τ_1, and μ_2 is a function of μ_1. But as τ_1 and τ_2 are solutions of the point equation of $\bar{S}$, τ_2 is at most a linear function of τ_1. The same is true of μ_2 and μ_1, since they satisfy the tangential equation of $\bar{S}$. On account of the linear character of equations (28) we have in all generality

$$\tau_2 = \tau_1 + \tau', \qquad \mu_2 = \mu_1 + \mu', \qquad m_2 = m_1 + m', \tag{91}$$

where τ', μ', and m' are constants. It follows at once from (28) that

$$\eta_2 = \eta_1, \qquad \beta_2 = \beta_1. \tag{92}$$

From (31) we have also

$$\theta_2 - \theta_1 = \mu' t - \tau'. \tag{93}$$

This equation is consistent with the equations obtained by replacing the quantities in (92) by the expressions (35) and similar ones for η_2 and β_2.

An exception to the preceding results arises in the case when $\bar{S}$ is a surface of revolution. In this case C coincides with $\bar{S}$ and ρ is a function of a single variable, say u. Now the point equation is satisfied by any function of u and consequently we are not justified in saying that τ_2 is necessarily linear in τ_1. As a matter to fact it is an easy matter to construct a large number of surfaces of revolution in relations of transformations of Ribaucour to the given surface, and the meridian planes are the circle planes. Hence we exclude from future consideration the case when $\bar{S}$ is a surface of revolution. It is readily shown that this is the only case for which τ_1 is a function of u or v alone.

From equations (42) and analogous ones in η_2 and β_2 we get

$$\begin{aligned} (\Phi_{20} - \Phi_{10}) m_0 \lambda_0 &= T_0^2 (m_2 \lambda_2 - m_1 \lambda_1), \\ (\Phi_{20} - \Phi_{10}) m_0 \theta_0 &= T_0^2 (m_2 \theta_2 - m_1 \theta_1), \end{aligned} \tag{94}$$

where Φ_{20} is given by (23) for $i = 2, j = 0$. In like manner from (37) and

analogous equations we have in consequence of (94)

$$(95)\quad \begin{aligned} (a_2 - a_1)\cos\omega - (b_2 - b_1)\sin\omega &= \frac{T_0}{m_0\lambda_0}(m_1\lambda_1 - m_2\lambda_2)\frac{1}{\sqrt{E}}\frac{\partial t}{\partial u}, \\ (a_2 - a_1)\cos\omega + (b_2 - b_1)\sin\omega &= \frac{T_0}{m_0\lambda_0}(m_1\lambda_1 - m_2\lambda_2)\frac{1}{\sqrt{G}}\frac{\partial t}{\partial v}. \end{aligned}$$

From [94] and [15] we have

$$(96)\quad \begin{aligned} \frac{\partial\Phi_{i0}}{\partial u} &= m_i(\lambda_i - \rho\theta_i)\,T_0\frac{\partial t}{\partial u} - m_0(\lambda_0 - \rho\theta_0)\sqrt{E}\,(a_i\cos\omega - b_i\sin\omega), \\ \frac{\partial\Phi_{i0}}{\partial v} &= m_i(\lambda_i + \rho\theta_i)\,T_0\frac{\partial t}{\partial v} - m_0(\lambda_0 + \rho\theta_0)\sqrt{G}\,(a_i\cos\omega + b_i\sin\omega), \end{aligned}$$

for $i = 1, 2$. Making use of these equations, we obtain from (94) by differentiation

$$\begin{aligned} \frac{\partial}{\partial u}(\Phi_{20} - \Phi_{10}) &= 2(\Phi_{20} - \Phi_{10})\frac{m_0(\lambda_0 - \rho\theta_0)}{T_0}\frac{\partial t}{\partial u}, \\ \frac{\partial}{\partial v}(\Phi_{20} - \Phi_{10}) &= 2(\Phi_{20} - \Phi_{10})\frac{m_0(\lambda_0 + \rho\theta_0)}{T_0}\frac{\partial t}{\partial v}. \end{aligned}$$

In consequence of (4) these equations are equivalent to

$$(97)\quad \Phi_{20} - \Phi_{10} = kT_0,$$

where k is a constant. Hence equations (94) become

$$(98)\quad m_2\theta_2 - m_1\theta_1 = km_0\theta_0, \qquad m_2\lambda_2 - m_1\lambda_1 = km_0\lambda_0.$$

If m_1 and m_2 are not equal, it follows from (93) and (98) that θ_1 is a linear function of θ_0 and t. If m_1 and m_2 are equal, θ_0 must be a linear function of t. Hence we have

THEOREM 4. *In order that two transformations A_{m_1} and A_{m_2} have the same circle-planes, it is necessary that θ_1 be a linear function of θ_0 and t; if $m_1 = m_2$, θ_0 must be a linear function of t.*

We postpone the consideration of the latter case to § 20.

9. When θ_1 is linear in θ_0 and t

Since θ_1 for a transformation K_m is determined only to within a constant factor, we take in all generality

$$(99)\quad \theta_1 = \theta_0 + c_1 t + d_1,$$

where c_1 and d_1 are constants. When this value is substituted in equations

(36), we get

$$\sqrt{E}\,(\cos\omega\, a_1 - \sin\omega\, b_1) = R_1\frac{\partial t}{\partial u} - \frac{\partial \theta_0}{\partial u},$$

(100)

$$\sqrt{G}\,(\cos\omega\, a_1 + \sin\omega\, b_1) = R_1\frac{\partial t}{\partial v} - \frac{\partial \theta_0}{\partial v},$$

where we have put

(101) $$R_1 = m_1\lambda_1(t_1 - t) - c_1.$$

From the equations

(102) $$\frac{\partial \lambda_i}{\partial u} = -\rho\frac{\partial \theta_i}{\partial u}, \qquad \frac{\partial \lambda_i}{\partial v} = \rho\frac{\partial \theta_i}{\partial v},$$

for $i = 1$, we have by integration

(103) $$\lambda_1 = \lambda_0 + c_1 r + e_1,$$

where e_1 is a constant and r is defined by

(104) $$\frac{\partial r}{\partial u} = -\rho\frac{\partial t}{\partial u}, \qquad \frac{\partial r}{\partial v} = \rho\frac{\partial t}{\partial v}.$$

Differentiating equation (101) and making use of equations (38) for $i = 1$, we obtain

(105) $$\frac{\partial R_1}{\partial u} = -m_1(\lambda_1 - \rho\theta_1)\frac{\partial t}{\partial u}, \qquad \frac{\partial R_1}{\partial v} = -m_1(\lambda_1 + \rho\theta_1)\frac{\partial t}{\partial v}.$$

If the above expressions for θ_1 and λ_1 be substituted in these equations, they can be integrated with the result

(106) $$R_1 = -\left(m_1\frac{T_0}{m_0} + Crt + Et + Dr + F\right),$$

where we have put

(107) $$C = m_1 c_1, \qquad E = m_1 e_1, \qquad D = m_1 d_1, \qquad F = m_1 f_1,$$

f_1 being the constant of integration.

We consider now the second transformation. We put

(108) $$\theta_2 = \theta_0 + c_2 t + d_2, \qquad \lambda_2 = \lambda_0 + c_2 r + e_2$$

without loss of generality. Hence from (98) and (107) we have

(109) $$C = m_2 c_2, \qquad E = m_2 e_2, \qquad D = m_2 d_2, \qquad m_2 - m_1 = km_0.$$

The equations analogous to (100) are

$$\sqrt{E}\,(\cos\omega\, a_2 - \sin\omega\, b_2) = R_2\frac{\partial t}{\partial u} - \frac{\partial \theta_0}{\partial u},$$

$$\sqrt{G}\,(\cos\omega\, a_2 + \sin\omega\, b_2) = R_2\frac{\partial t}{\partial v} - \frac{\partial \theta_0}{\partial v},$$

where

$$R_2 = m_2 \lambda_2 (t_2 - t) - c_2 .$$

In order that these equations and (100) be consistent with (95), we must have

$$R_2 - R_1 = \frac{T_0}{m_0} (m_1 - m_2) . \tag{110}$$

In consequence of this equation, (107), and (109), it follows that in the expression for R_2 similar to (106) we must have the same constant F, that is we may write

$$F = m_2 f_2 . \tag{111}$$

Again from (95) we have with the aid of (10), (98), and (109) the first two of the following:

$$\begin{gathered} a_2 - a_1 = \frac{m_2 - m_1}{m_0} a_0 , \qquad b_2 - b_1 = \frac{m_2 - m_1}{m_0} b_0 , \\ w_2 - w_1 = \frac{m_2 - m_1}{m_0} w_0 . \end{gathered} \tag{112}$$

The last one follows when we express the condition that the functions a_i and b_i satisfy the fundamental equations [10] of a transformation K, namely

$$\begin{aligned} \frac{\partial a_i}{\partial u} &= - m_i (\lambda_i - \rho\theta_i) \sqrt{E} \cos \omega + b_i A + w_i D/2 \sqrt{E} \cos \omega , \\ \frac{\partial a_i}{\partial v} &= - m_i (\lambda_i + \rho\theta_i) \sqrt{G} \cos \omega - b_i B + w_i D''/2 \sqrt{G} \cos \omega , \\ \frac{\partial b_i}{\partial u} &= m_i (\lambda_i - \rho\theta_i) \sqrt{E} \sin \omega - a_i A - w_i D/2 \sqrt{E} \sin \omega , \\ \frac{\partial b_i}{\partial v} &= - m_i (\lambda_i + \rho\theta_i) \sqrt{G} \cos \omega + a_i B + w_i D''/2 \sqrt{G} \sin \omega , \\ \frac{\partial w_i}{\partial u} &= - \frac{D}{2\sqrt{E}} \left(\frac{a_i}{\cos \omega} - \frac{b_i}{\sin \omega} \right), \\ \frac{\partial w_i}{\partial v} &= - \frac{D''}{2\sqrt{G}} \left(\frac{a_i}{\cos \omega} + \frac{b_i}{\sin \omega} \right), \end{aligned} \tag{113}$$

where

$$A = \sqrt{\frac{E}{G}} \frac{\partial \log \sqrt{\rho}}{\partial v} \sin 2\omega - \frac{\partial \omega}{\partial u}, \qquad B = \sqrt{\frac{G}{E}} \frac{\partial \log \sqrt{\rho}}{\partial u} \sin 2\omega - \frac{\partial \omega}{\partial v} . \tag{114}$$

We recall from M_2, § 6, that a set of functions a_i, b_i, w_i, θ_i, λ_i, t_i satisfying

(36), (38), (102), (113), and the relation

$$2m_i\,\theta_i\lambda_i - (a_i^2 + b_i^2 + w_i^2) + m_i^2\lambda_i^2(t_i - t)^2 = 0, \tag{115}$$

determine a transformation K_m of surfaces C and a transformation A_m of surfaces Ω.

10. Transformations A_m with θ_1 linear in θ_0 and t

We consider now the transformation for which θ_1 has the form (99). From equations (100) we have, in consequence of (10),

$$\begin{aligned} a_1 &= -R_1\frac{a_0}{T_0} - \frac{1}{2\cos\omega}\left(\frac{1}{\sqrt{E}}\frac{\partial\theta_0}{\partial u} + \frac{1}{\sqrt{G}}\frac{\partial\theta_0}{\partial v}\right), \\ b_1 &= -R_1\frac{b_0}{T_0} - \frac{1}{2\sin\omega}\left(-\frac{1}{\sqrt{E}}\frac{\partial\theta_0}{\partial u} + \frac{1}{\sqrt{G}}\frac{\partial\theta_0}{\partial v}\right). \end{aligned} \tag{116}$$

When these expressions are substituted in the first four of equations (113) for $i = 1$, the resulting equations reduce to the following two:

$$\begin{aligned} &D\left(w_1 + \frac{w_0}{T_0}R_1\right) + (\theta_0)_{11} \\ &\qquad - \frac{1}{\rho}\left[m_1(\lambda_1 - \rho\theta_1) + \frac{m_0 R_1}{T_0}(\lambda_0 - \rho\theta_0)\right] = 0, \\ &D''\left(w_1 + \frac{w_0}{T_0}R_1\right) + (\theta_0)_{22} \\ &\qquad - \frac{1}{\rho}\left[m_1(\lambda_1 + \rho\theta_1) + \frac{m_0 R_1}{T_0}(\lambda_0 + \rho\theta_0)\right] = 0, \end{aligned} \tag{117}$$

where we have put

$$\begin{aligned} (\theta_0)_{11} &= \frac{\partial^2\theta_0}{\partial u^2} - \left\{{11 \atop 1}\right\}\frac{\partial\theta_0}{\partial u} - \left\{{11 \atop 2}\right\}\frac{\partial\theta_0}{\partial v}, \\ (\theta_0)_{22} &= \frac{\partial^2\theta_0}{\partial v^2} - \left\{{22 \atop 1}\right\}\frac{\partial\theta_0}{\partial u} - \left\{{22 \atop 2}\right\}\frac{\partial\theta_0}{\partial v}, \end{aligned} \tag{118}$$

the Christoffel symbols $\left\{{rs \atop t}\right\}$ being formed with respect to the linear element of C; their expressions are given by M$_1$ (108).

In like manner the last two of equations (113) become

$$\begin{aligned} \frac{\partial w_1}{\partial u} &= -\frac{R_1}{T_0}\frac{\partial w_0}{\partial u} + \frac{D}{H^2}\left(G\frac{\partial\theta_0}{\partial u} - F\frac{\partial\theta_0}{\partial v}\right), \\ \frac{\partial w_1}{\partial v} &= -\frac{R_1}{T_0}\frac{\partial w_0}{\partial v} + \frac{D''}{H^2}\left(E\frac{\partial\theta_0}{\partial v} - F\frac{\partial\theta_0}{\partial u}\right). \end{aligned} \tag{119}$$

If the first of equations (117) be differentiated with respect to v and the second with respect to u, and the derivatives of w_1 be replaced by their ex-

pressions from (119), the resulting equations vanish identically because of the Codazzi and Gauss equations* for C. Hence it is only necessary to express the condition that the expressions for w_1 given by (117) shall be equal. This is

$$
(120)\qquad \begin{aligned} \rho\,[D''(\theta_0)_{11} - D(\theta_0)_{22}] = D''\left[m_1(\lambda_1 - \rho\theta_1) + \frac{R_1}{T_0} m_0(\lambda_0 - \rho\theta_0)\right] \\ - D\left[m_1(\lambda_1 + \rho\theta_1) + \frac{R_1}{T_0} m_0(\lambda_0 + \rho\theta_0)\right]. \end{aligned}
$$

From (101) and (106) we have

$$
(121)\qquad m_1\lambda_1(t_1 - t) = c_1 - \left(\frac{T_0}{m_0} m_1 + Crt + Et + Dr + F\right).
$$

It is readily shown that this value of t_1 satisfies equations (38) for $i = 1$.

It can be shown that if $a_i, b_i, w_i, \lambda_i, \theta_i, t_i$ satisfy equations (36), (38), (102), and (113), the left-hand member of equation (115) is constant. As we desire a transformation K_m, we must show that for the above functions this constant is zero. In order to do this we must simplify the expression for w_1.

By definition the differential parameter $\Delta_1(\theta_0, t)$ formed with respect to the linear element of any surface is given by

$$
\Delta_1(\theta_0, t) = \frac{G\dfrac{\partial\theta_0}{\partial u}\dfrac{\partial t}{\partial u} - F\left(\dfrac{\partial\theta_0}{\partial u}\dfrac{\partial t}{\partial v} + \dfrac{\partial\theta_0}{\partial v}\dfrac{\partial t}{\partial u}\right) + E\dfrac{\partial\theta_0}{\partial v}\dfrac{\partial t}{\partial v}}{EG - F^2}.
$$

By differentiation we have

$$
(122)\qquad \begin{aligned} \frac{\partial}{\partial u}\Delta_1(\theta_0, t) &= \frac{G\dfrac{\partial t}{\partial u} - F\dfrac{\partial t}{\partial v}}{EG - F^2}(\theta_0)_{11} + \frac{G\dfrac{\partial\theta_0}{\partial u} - F\dfrac{\partial\theta_0}{\partial v}}{EG - F^2}(t)_{11}, \\ \frac{\partial}{\partial v}\Delta_1(\theta_0, t) &= \frac{E\dfrac{\partial t}{\partial v} - F\dfrac{\partial t}{\partial u}}{EG - F^2}(\theta_0)_{22} + \frac{E\dfrac{\partial\theta_0}{\partial v} - F\dfrac{\partial\theta_0}{\partial u}}{EG - F^2}(t)_{22}, \end{aligned}
$$

where the expressions for $(t)_{11}$ and $(t)_{22}$ are analogous to those for $(\theta_0)_{11}$ and $(\theta_0)_{22}$ given by (118).

For a surface C, in consequence of (2), we have

$$
(123)\qquad \begin{gathered} G\frac{\partial t}{\partial u} - F\frac{\partial t}{\partial v} = \frac{1}{\rho}\frac{\partial t}{\partial u}, \qquad E\frac{\partial t}{\partial v} - F\frac{\partial t}{\partial u} = \frac{1}{\rho}\frac{\partial t}{\partial v}, \\ \Delta_1(\theta_0, t) = \frac{\dfrac{\partial\theta_0}{\partial u}\dfrac{\partial t}{\partial u} + \dfrac{\partial\theta_0}{\partial v}\dfrac{\partial t}{\partial v}}{\rho H^2}. \end{gathered}
$$

* E., p. 155.

Moreover, from [43],

$$(124)\qquad \begin{aligned} (t)_{11} &= -\frac{w_0 D}{T_0} + \frac{m_0}{\rho T_0}(\lambda_0 - \rho\theta_0), \\ (t)_{22} &= -\frac{w_0 D''}{T_0} + \frac{m_0}{\rho T_0}(\lambda_0 + \rho\theta_0). \end{aligned}$$

Hence for surfaces C we have from (122)

$$(125)\qquad \begin{aligned} \frac{1}{\rho H^2}\frac{\partial t}{\partial u}(\theta_0)_{11} &= \frac{\partial}{\partial u}\Delta_1(\theta_0, t) + \frac{G\frac{\partial\theta_0}{\partial u} - F\frac{\partial\theta_0}{\partial v}}{H^2 T_0}[w_0 D - m_0(\lambda_0 - \rho\theta_0)], \\ \frac{1}{\rho H^2}\frac{\partial t}{\partial v}(\theta_0)_{22} &= \frac{\partial}{\partial v}\Delta_1(\theta_0, t) + \frac{E\frac{\partial\theta_0}{\partial v} - F\frac{\partial\theta_0}{\partial u}}{H^2 T_0}[w_0 D'' - m_0(\lambda_0 + \rho\theta_0)]. \end{aligned}$$

With the aid of these results, (117), and [48], namely

$$(126)\qquad \frac{\partial}{\partial u}\log w_0 = \frac{D}{H}\frac{\partial t}{\partial u}, \qquad \frac{\partial}{\partial v}\log w_0 = \frac{D''}{H}\frac{\partial t}{\partial v},$$

equations (119) may be written

$$(127)\qquad \begin{aligned} &\frac{\partial}{\partial u}\left[w_0\left(w_1 + \frac{R_1}{T_0}w_0\right)\right] \\ &\qquad = -T_0\frac{\partial}{\partial u}\Delta_1(\theta_0, t) + \frac{G\frac{\partial\theta_0}{\partial u} - F\frac{\partial\theta_0}{\partial v}}{H^2\rho} m_0(\lambda_0 - \rho\theta_0), \\ &\frac{\partial}{\partial v}\left[w_0\left(w_1 + \frac{R_1}{T_0}w_0\right)\right] \\ &\qquad = -T_0\frac{\partial}{\partial v}\Delta_1(\theta_0, t) + \frac{E\frac{\partial\theta_0}{\partial v} - F\frac{\partial\theta_0}{\partial u}}{H^2\rho} m_0(\lambda_0 + \rho\theta_0). \end{aligned}$$

In consequence of (4) and (123) these equations may be integrated in the form

$$(128)\qquad w_0\left(w_1 + \frac{R_1}{T_0}w_0\right) = m_0(\lambda_0\theta_0 + g) - T_0\Delta_1(\theta_0, t),$$

where g is a constant. From (110) and (112) it follows that g does not change in passing from the transformation of $\bar{S}$ into $\bar{S}_1$ to that of $\bar{S}$ into $\bar{S}_2$; the importance of this remark will be seen later.

11. Special surfaces Ω

When the values of a_1, b_1, w_1, t_1, as given by (116), (128), and (121) are substituted in equation (115) for $i = 1$, we get

$$
\begin{aligned}
(129)\quad & \Delta_1\theta_0 + H^2\rho^2[\Delta_1(\theta_0, t) - N]^2 + 2C\left(Nrt + \frac{T_0}{m_0} - \theta_0 r - \lambda_0 t\right) \\
& \qquad + 2D(rN - \lambda_0) + 2E(tN - \theta_0) + 2FN + G = 0,
\end{aligned}
$$

where

$$
(130)\qquad N = \frac{m_0(\lambda_0\theta_0 + g)}{T_0}, \qquad \Delta_1\theta_0 = \Delta_1(\theta_0, \theta_0),
$$

and G is a constant such that

$$
(131)\qquad G = 2m_1 g + \frac{2CF}{m_1} - \frac{2DE}{m_1} - \frac{C^2}{m_1^2}.
$$

When equation (129) is differentiated with respect to u and v separately, the resulting equations are reducible to*

$$
(132)\quad
\begin{aligned}
& \left(\frac{\partial\theta_0}{\partial u} - N\frac{\partial t}{\partial u}\right)\Big[(\theta_0)_{11} + \frac{1}{\rho}(CJ_1 + DK_1 + EL_1 + FM_1) \\
& \qquad\qquad + \frac{D}{w_0}T_0(N - \Delta_1(\theta_0, t))\Big] = 0, \\
& \left(\frac{\partial\theta_0}{\partial v} - N\frac{\partial t}{\partial v}\right)\Big[(\theta_0)_{22} + \frac{1}{\rho}(CJ_2 + DK_2 + EL_2 + FM_2) \\
& \qquad\qquad + \frac{D''}{w_0}T_0(N - \Delta_1(\theta_0, t))\Big] = 0,
\end{aligned}
$$

where

$$
(133)\quad
\begin{aligned}
& M_i = \frac{m_0(\lambda_0 \mp \rho\theta_0)}{T_0}, \qquad J_i = M_i rt - r \pm \rho t, \\
& K_i = M_i r \pm \rho, \qquad L_i = M_i t - 1,
\end{aligned}
$$

with the upper or lower sign according as i is 1 or 2. We consider now the various ways in which (132) can be satisfied.

If the first terms of (132) are zero, there is a functional relation between θ_0 and t. Since both are solutions of equation (1), this relation must be linear. Later (§ 17) we show that when θ_0 is a linear function of t, these terms are not equal to zero.

We consider next the case

$$
\frac{\partial\theta_0}{\partial u} = N\frac{\partial t}{\partial u}, \qquad \frac{\partial\theta_0}{\partial v} \mp N\frac{\partial t}{\partial v},
$$

$$
(134)\quad (\theta_0)_{22} + \frac{1}{\rho}(CJ_2 + DK_2 + EL_2 + FM_2) + \frac{D''}{w_0}T_0(N - \Delta_1(\theta_0, t)) = 0.
$$

* It should be noted that D which multiplies the last expression in the first equation is one of the fundamental functions for C and not a constant, as the other D is.

If the first of (134) be multiplied by $m_0(\lambda_0 - \rho\theta_0)$, it can be integrated, in consequence of (4), in the form

$$(135) \qquad m_0(\lambda_0 \theta_0 + g) = T_0 V,$$

that is, N is equal to V, which is a function of v alone.

Accordingly from the first of (134), (104), and (5) we have

$$(136) \qquad \theta_0 = Vt + V_1, \qquad \lambda_0 = Vr + V_2,$$

where V_1 and V_2 are functions of v alone.

On the assumption that these quantities satisfy the second of (5) we get

$$V' r + V_2' = \rho(V' t + V_1'),$$

where the primes indicate differentiation with respect to v. Since this value of r must satisfy equations (104), we must have

$$(137) \quad (V' t + V_1')\sqrt{\rho} = V_3, \qquad 2\frac{\partial \sqrt{\rho}}{\partial v}\frac{V_3}{V'} + \rho\left(\frac{V_1'}{V'}\right)' - \left(\frac{V_2'}{V'}\right)' = 0,$$

where V_3 is a function of v given by

$$V_3^2 = V_1' V_2' - V'\left(\frac{V_1 V_2 + g}{V}\right)'.$$

In order to obtain this expression for V_3, we solved the above equations for r and t, getting

$$(138) \qquad r = \frac{1}{V'}(\sqrt{\rho}V_3 - V_2'), \qquad t = \frac{1}{V'}\left(\frac{V_3}{\sqrt{\rho}} - V_1'\right),$$

and substituted in the equation obtained by differentiating (135) with respect to v.

The remaining conditions to be satisfied are the last of (134) and (6). From the former of these we obtain an equation of the form

$$\frac{\partial \sqrt{\rho}}{\partial u} = P,$$

where P is a determinate function of V, V_1, V_2, their derivatives and $\sqrt{\rho}$, but the function is very complicated. However, in any particular case it would be possible to see whether it, (136), and (6) are consistent. Accordingly in what follows we make exception of the cases where θ_0 is of the form (136) or with V and V_1 replaced by U and U_1, functions of u alone.

When we equate to zero the second terms of the two equations (132), the resulting equations are reducible by means of (128) to (117). Hence, when C satisfies (129), the values of a_1, b_1, w_1, t_1, as given by (116), (128), and (121), determine a transformation K_{m_1} of C. Now m_1 is a solution of the cubic (131).

Moreover, each solution of this cubic determines a transformation of C into a surface C_i, and, as follows from § 9 the transformations A_m so determined have the same circle-planes.

From (18), (98), (109), and (112) we have

$$m_2 \lambda_2 x_2 - m_1 \lambda_1 x_1 + (m_2 - m_1) \lambda_0 x_0 = 0.$$

Hence if M_1, M_2, M_3 are the points on the three transforms of M in the above special case, it follows that M_1, M_2, M_3 are collinear and their line passes through M_0. We have seen (§ 8) that if m_2 and m_1 are equal, θ_0 must be a linear function of t. Hence for the present we exclude the case where two of the roots of (131) are equal (cf. § 20). This imposes a condition on the constants in equation (129), which will be assumed in what follows.

We shall say that two surfaces C whose functions satisfy the same equation (129) are *special surfaces C of the same class.* Later (§ 12) it will be shown that if C is a special surface, C_0 is a special surface of the same class. Hence we speak of $\bar{S}$ as a *special surface* Ω of the class determined by the constants C, D, E, F, G, and g. Each of the three surfaces C_i determines a transform $\bar{S}_i$ of $\bar{S}$. We speak of these particular surfaces C_i and $\bar{S}_i$ as the *complementary transforms* of C and $\bar{S}$ respectively. The preceding results may be stated in

THEOREM 5. *If C is a special surface satisfying* (129) *in which the constants C, D, E, F, G, g are such that the roots of* (131) *are real and distinct, and θ_0 is not a linear function of t with coefficients which are functions of the same single variable or constants, there can be found directly three transforms of C such that corresponding points on these surfaces are collinear. These three surfaces C_i determine three transformations A_{m_i} of the surface $\bar{S}$, associated with C and C_0, such that the circle-planes of the three transformations coincide.*

The investigations of the last two sections enable us to state the following theorem which is fundamental in what follows:

THEOREM 6. *When a surface C admits a transformation K_m into another surface C_1, such that the function θ_1 is linear in t and θ_0 with constant coefficients, the surface C is a special surface, and C_1 is a complementary transform of C.*

12. Complementary transformations of special surfaces Ω

Having thus established the existence of special surfaces Ω and transformations of them for which θ_1 is a linear function of t and θ_0, we consider in this section the surfaces $\bar{S}_i$ resulting from these transformations.

From (116) and (128) we get

$$\Phi_{10} = \sum a_1 a_0 = - R_1 T_0 + m_0 (\lambda_0 \theta_0 + g).$$

Hence from (25), (31), (33), and (35) follow, in consequence of (99) and (101),

$$(139)\quad \begin{gathered} \mu_1 = c_1 + N, \qquad \tau_1 = Nt - \theta_0 - d_1, \qquad \tau_1\sigma_1 = \mu_1\frac{(R_1 + c_1)}{m_1} + \lambda_1\theta_1, \\ \eta_1 = \sqrt{\rho}\left(N\frac{\partial t}{\partial u} - \frac{\partial \theta_0}{\partial u}\right), \qquad \beta_1 = \sqrt{\rho}\left(N\frac{\partial t}{\partial v} - \frac{\partial \theta_0}{\partial v}\right). \end{gathered}$$

Equation [93] is reducible to

$$m_0\lambda_0(\theta_1 - \theta_{01}) = T_0(c_1 + N) = T_0\mu_1.$$

Hence in consequence of the identities (16), we have

$$\lambda_1\theta_{10} = -\lambda_0\theta_1 + \frac{T_0}{m_0}\mu_1, \qquad \theta_1\lambda_{10} = -\theta_0\lambda_1 + \frac{T_0}{m_0}\mu_1,$$

which are reducible to

$$(140)\quad \begin{aligned} \lambda_1\theta_{10} &= -\lambda_0(c_1 t + d_1) + \frac{T_0}{m_0}c_1 + g, \\ \theta_1\lambda_{10} &= -\theta_0(c_1 r + e_1) + \frac{T_0}{m_0}c_1 + g. \end{aligned}$$

It is readily shown that these functions satisfy equations for $\bar{S}_1$ analogous to (5), namely

$$(141)\quad \frac{\partial\lambda_{10}}{\partial u} = -\frac{\lambda_1^2}{\rho\theta_1^2}\frac{\partial\theta_{10}}{\partial u}, \qquad \frac{\partial\lambda_{10}}{\partial v} = \frac{\lambda_1^2}{\rho\theta_1^2}\frac{\partial\theta_{10}}{\partial v}.$$

If θ_1^{-1} and λ_1^{-1} denote the functions by means of which we pass from $\bar{S}_1$ to $\bar{S}$, and if we show that

$$\theta_1^{-1} = \theta_{10} + c_1' t_1 + d_1',$$

then we shall have shown that $\bar{S}_1$ also is a special surface Ω.

We recall from M_2, § 1, that the fundamental equations of a transformation K of a surface C are

$$(142)\quad \begin{aligned} \frac{\partial}{\partial u}(\lambda_1 x_1) &= -\rho\left(x\frac{\partial\theta_1}{\partial u} - \theta_1\frac{\partial x}{\partial u}\right), \\ \frac{\partial}{\partial v}(\lambda_1 x_1) &= \rho\left(x\frac{\partial\theta_1}{\partial v} - \theta_1\frac{\partial x}{\partial v}\right). \end{aligned}$$

For the inverse transformation from C_1 to C the analogous equations are

$$\begin{aligned} \frac{\partial}{\partial u}(\lambda_1^{-1} x) &= -\frac{\lambda_1^2}{\rho\theta_1^2}\left(x_1\frac{\partial\theta_1^{-1}}{\partial u} - \theta^{-1}\frac{\partial x_1}{\partial u}\right), \\ \frac{\partial}{\partial v}(\lambda_1^{-1} x) &= \frac{\lambda_1^2}{\rho\theta_1^2}\left(x_1\frac{\partial\theta_1^{-1}}{\partial v} - \theta_1^{-1}\frac{\partial x_1}{\partial v}\right). \end{aligned}$$

From these equations it is readily found that

$$(143)\qquad \theta_1^{-1} = \frac{\kappa}{\lambda_1}, \qquad \lambda_1^{-1} = \frac{\kappa}{\theta_1},$$

where κ denotes an arbitrary constant.

When this value and the expressions for θ_{10} and t_1 from (140) and (121) respectively are substituted in the above equation, we find that it vanishes identically, if we take

$$(144)\qquad c_1' = c_1, \qquad d_1' = d_1, \qquad \kappa = g - \frac{CF}{m_1^2} + \frac{DE}{m_1^2} + \frac{C^2}{m_1^3}.$$

Hence $\bar{S}_1$ is a special surface Ω. It remains to show that it is of the same class as $\bar{S}$.

We introduce for the surface C_1 a function r_1 analogous to r for C. From (104) and (141) it follows that

$$(145)\qquad \frac{\partial r_1}{\partial u} = -\frac{\lambda_1^2}{\rho\theta_1^2}\frac{\partial t_1}{\partial u}, \qquad \frac{\partial r_1}{\partial v} = \frac{\lambda_1^2}{\rho\theta_1^2}\frac{\partial t_1}{\partial v}.$$

As thus defined, r_1 is determined only to within an additive constant. But it is readily shown that this constant may be so chosen that we have

$$(146)\qquad \theta_1(r_1 - r) = \lambda_1(t_1 - t).$$

Since the functions λ_1^{-1} and θ_1^{-1} must satisfy equations (141), as well as λ_{10} and θ_{10}, we have

$$(147)\qquad \theta_1^{-1} = \theta_{10} + c_1 t_1 + d_1, \qquad \lambda_1^{-1} = \lambda_{10} + c_1 r_1 + e_1,$$

in consequence of (144) and (146).

From these results it follows that the equation analogous to (121) is

$$m_1 \lambda_1^{-1}(t - t_1) = c_1 - \left(\frac{T_{10}\, m_1}{m_0} + Cr_1 t_1 + Et_1 + Dr_1 + F'\right).$$

In consequence of (21), (16), (121), (140), (146), and (147) it follows that $F' = F$.

If the constant g is the same for $\bar{S}_1$ as for $\bar{S}$, the function μ_1^{-1} of the transformation from $\bar{S}_1$ to $\bar{S}$ must be

$$\mu_1^{-1} = c_1 + \frac{m_0}{T_{10}}(\lambda_{10}\,\theta_{10} + g),$$

as follows from (130) and (139). If μ_1^{-1} be replaced by its expression from (53), it is readily found that the preceding equation is true, when we take $c = \kappa$, as given by (144).

Since m_1 is the constant of the transformation from $\bar{S}_1$ to $\bar{S}$, from (131) it follows that G has the same value for $\bar{S}_1$ as for $\bar{S}$. Hence we have

THEOREM 7. *The complementary transforms of a special surface Ω are special surfaces of the same class.*

We consider now the four surfaces C, C_0, C_1, C_{10} which are associated with a pair of complementary surfaces $\bar{S}$ and $\bar{S}_1$. The functions θ_0^{-1} and λ_0^{-1}, by means of which C is obtained from C_0, are given by equations similar to (143), namely

$$\theta_0^{-1} = \kappa_0/\lambda_0, \qquad \lambda_0^{-1} = \kappa_0/\theta_0,$$

where κ_0 is a constant. From (140) and (16) we have

$$\theta_{01} = c_1 t + d_1 - \frac{T_0}{m_0 \lambda_0} c_1 - \frac{g}{\lambda_0}.$$

In consequence of [67], namely

$$T_0 = m_0 \lambda_0 (t - t_0), \tag{148}$$

it follows that if we take $\kappa_0 = -g$, we have

$$\theta_{01} = \theta_0^{-1} + c_1 t_0 + d_1. \tag{149}$$

On the assumption that r_0 is chosen so that a relation similar to (146) holds, we find from (16) and (149)

$$\lambda_{01} = \lambda_0^{-1} + c_1 r_0 + e_1. \tag{150}$$

For a *quatern* (cf. M_2, § 7) of surfaces C the functions t satisfy an equation similar to (17), namely

$$\lambda_{ij}\, \theta_i\, t_{ij} = \theta_{ij} \lambda_i t - \lambda_i \theta_j t_i + \lambda_j \theta_i t_j. \tag{151}$$

Also from (148) and an analogous equation we have

$$T_0^{-1} = m_0 \lambda_0^{-1} (t_0 - t) = -\lambda_0^{-1} T_0/\lambda_0.$$

With the aid of these equations it is readily shown that an equation analogous to (121), namely

$$m_1 \lambda_{01} (t_{10} - t_0) = c_1 - \left(\frac{T_0^{-1}}{m_0} m_1 + C r_0 t_0 + E t_0 + D r_0 + F \right), \tag{152}$$

is satisfied.

In accordance with the general theory of the permutability of transformations K_m, the constant m has the same value for the transformations from C into C_1 and C_0 into C_{10}. Hence all three constants m are the same for complementary transforms of C_0 as of C. In view of the above results g and G accordingly have the same values for C_0 as for C. Hence we have:

THEOREM 8. *When C is a special surface, the conjugate surface C_0, which with C determines a special surface Ω, is of the same class as C; if C_1 is a complementary transform of C, and C_{10} is the fourth surface of the quatern, then C_{10} is a special surface of the same class as C and is a complementary transform of C_0.*

13. General transformations of special surfaces

In this section we apply the results of the theorem of permutability of transformations A_m of surfaces Ω [§ 9] to show that if $\bar{S}$ and $\bar{S}_1$ are complementary surfaces Ω, it is possible to find surfaces $\bar{S}_2$ such that $\bar{S}_2$ and the surface $\bar{S}_{12}$ are complementary surfaces, where $\bar{S}_{12}$ is the surface forming with $\bar{S}$, $\bar{S}_1$, and $\bar{S}_2$ a quatern of surfaces Ω.

The function θ_{21} determining the transformation from $\bar{S}_2$ to $\bar{S}_{12}$ is given by [104], namely

$$\begin{aligned}\lambda_2\,\theta_{21}(m_2 - m_1) = m_2\,\lambda_2\,\theta_1 + m_1\,\lambda_1\,\theta_2 - \Phi_{12}& \\ + m_1\,m_2\,\lambda_1\,\lambda_2\,(t_1 - t)(t_2 - t)&.\end{aligned} \tag{153}$$

From the general theory of transformations A_m we know that on the normal to $\bar{S}_2$ there are two points M_2 and M_{20} which describe a surface C_2 and a conjugate surface C_{20}, and that the four surfaces C, C_0, C_2, C_{20} form a quatern. From [93] we have that the function θ_{20}, determining the transformation from C_2 to C_{20}, is given by

$$\lambda_2\,\theta_{20} = -\lambda_0\,\theta_2 + \frac{T_0}{m_0}\,m_2\,\lambda_2\,(t_2 - t) + \frac{\Phi_{20}}{m_0}. \tag{154}$$

When these expressions are substituted in

$$\theta_{21} = \theta_{20} + c_1\,t_2 + d_1, \tag{155}$$

we get

$$\begin{aligned}(Cr + E + m_2\lambda_0)\,\theta_2 + (Ct_2 + D + m_2\,\theta_0)\,\lambda_2 - \Phi_{12} + \frac{m_1 - m_2}{m_0}\,\Phi_{20}& \\ - m_2\,\lambda_2\,(t_2 - t)\left(Crt + Dr + Et + F + m_2\frac{T_0}{m_0}\right) = 0&.\end{aligned} \tag{156}$$

If the left-hand member of this equation be differentiated with respect to u and v separately, and it be required merely that θ_2 satisfy equations analogous to (36), the resulting equations vanish identically. Hence for all transformations A_{m_2} of $\bar{S}$ the left-hand member is a constant. Consequently of the ∞^r transformations A_{m_2} of $\bar{S}$ into $\bar{S}_2$, ∞^{r-1} are such that (155) is satisfied, and therefore, by Theorem 6, $\bar{S}_{12}$ is a complementary transform of $\bar{S}_2$. Hence $\bar{S}_2$ and $\bar{S}_{12}$ are special surfaces.

By methods analogous to those followed in the preceding section it can be shown that

$$\lambda_{21} = \lambda_{20} + c_1\,r_2 + e_1,$$

$$m_1\,\lambda_{21}\,(t_{21} - t_2) = c_1 - \left(T_{20}\frac{m_1}{m_0} + Cr_2\,t_2 + Et_2 + Dr_2 + F\right).$$

Analogously to (30) the function μ_{21} of the transformation from $\bar{S}_2$ to $\bar{S}_{12}$

is of the form

$$\mu_{21} = \frac{m_0 [\lambda_{21} (\theta_{21})_0 + \lambda_{20} \theta_{21}]}{T_{20}},$$

where $(\theta_{21})_0$ is the function determining the conjugate $(C_{21})_0$ of C_{21} such that C_2, C_{20}, C_{21}, $(C_{21})_0$ form a quatern. From equations analogous to [84] we have

$$(157) \qquad (\theta_{21})_0 \lambda_{21} \theta_2 = - \theta_{20} \lambda_2 \theta_1 + \theta_{21} \lambda_2 \theta_0 + \theta_{10} \lambda_1 \theta_2 .$$

By means of these results we find that

$$\mu_{21} = c_1 + \frac{m_0 (\theta_{20} \lambda_{20} + g)}{T_{20}}.$$

As the transformation constant from C_2 to C_{12} is m_1, it follows that G is the same as for C. Hence:

THEOREM 9. *If $\bar{S}$ is a special surface Ω and $\bar{S}_1$ is a complementary transform, and $\bar{S}_2$ is obtained from $\bar{S}$ by a transformation A_{m_2} whose functions satisfy* (156), *$\bar{S}_2$ is a special surface of the same class, and $\bar{S}_{12}$, which is the fourth surface of the quatern determined by $\bar{S}$, $\bar{S}_1$, and $\bar{S}_2$, is a complementary transform of $\bar{S}_2$.*

If in the expression for Φ_{12}, as given by (23), we replace a_1, b_1, w_1, by their values from (116) and (128), then in consequence of (123), and equations analogous to (121), (42′), and (25), the equation (156) is reducible to

$$(158) \qquad \begin{aligned} &(Cr + E + m_2 \lambda_0) \theta_2 + (Ct + D + m_2 \theta_0) \lambda_2 \\ &\quad + \sqrt{\rho} \left(\eta_2 \frac{\partial \theta_0}{\partial u} + \beta_2 \frac{\partial \theta_0}{\partial v} \right) - N \left[\frac{\Phi_{20}}{T_0} + \sqrt{\rho} \left(\eta_2 \frac{\partial t}{\partial u} + \beta_2 \frac{\partial t}{\partial v} \right) \right] \\ &\quad - \mu_2 \left(Crt + Et + Dr + F + \frac{m_2 T_0}{m_0} \right) = 0 . \end{aligned}$$

From the preceding considerations we know that the left-hand member of this equation equated to a constant is a first integral of equations (28).

Since m_1 does not appear in equation (158), it follows that m_2 can take on the value m_1 as well as all other values. Hence we have

THEOREM 10. *When a special surface $\bar{S}$ is subjected to a transformation A_m whose functions satisfy equation* (158), *the new surface is a special surface of the same class.*

If C, C_i, C_j, C_{ij} form a quatern under general transformations K, the transformation function θ_{ij} is given by [72], namely

$$(159) \qquad \begin{aligned} \frac{\partial}{\partial u} (\lambda_i \theta_{ij}) &= - \rho \left(\theta_i \frac{\partial \theta_j}{\partial u} - \theta_j \frac{\partial \theta_i}{\partial u} \right), \\ \frac{\partial}{\partial v} (\lambda_i \theta_{ij}) &= \rho \left(\theta_i \frac{\partial \theta_j}{\partial v} - \theta_j \frac{\partial \theta_i}{\partial v} \right). \end{aligned}$$

When θ_1 is of the form (99), and θ_2 is any other solution of equation (1), it is readily found that the expression (155) for θ_{21} satisfies these equations for $i = 2, j = 1$, if θ_{20} satisfies them for $i = 2, j = 0$. Hence if C and C_1 are complementary surfaces and C_2 is any transform of C, one of the infinity of surfaces S', which with C, C_1, C_2 forms a quatern under general transformations K, is determined by the value (155) of θ_{21}. But one, and only one, of the surfaces S' is a surface C (cf. [§ 9]), say C_{12}, in the relation of transformations K'_{m_2} and K'_{m_1} with C_1 and C_2 respectively. Moreover, we have seen that if the transformation K_{m_2} from C to C_2 satisfies (156), the function θ_{21} determining K'_{m_1} is of the form (155). Hence we have

THEOREM 11. *If C and C_1 are complementary surfaces for which $\theta_1 = \theta_0 + c_1 t + d_1$ and C_2 is any transform of C, the surface arising from C_2 by the transformation K determined by* (155) *is a surface C, if and only if* (156) *is satisfied; in this case it is complementary to C_2.*

14. THEOREM OF PERMUTABILITY OF GENERAL TRANSFORMATIONS OF SPECIAL SURFACES Ω

Let C_2 and C_3 be two surfaces obtained from a special surface C by transformations K_{m_2} and K_{m_3} which satisfy equations of the form (156), and let C_{23} be the unique surface C which with C, C_2, and C_3 forms a quatern of surfaces C. Let C_1 be the surface complementary to C determined by the constant m_1, and let C_{12} and C_{13} be the surfaces complementary to C_2 and C_3 in accordance with Theorem 11.

The functions θ_{21}, θ_{23}, θ_{31}, determining the transformations of C_2 into C_{12} and C_{23} and C_3 into C_{13}, are given by equations analogous to (153), namely

$$\begin{aligned}\lambda_i \theta_{ij}(m_i - m_j) = m_i \lambda_i \theta_j + m_j \lambda_j \theta_i - \Phi_{ij} \\ + m_i m_j \lambda_i \lambda_j (t_i - t)(t_j - t)\end{aligned} \quad \begin{pmatrix} i = 2, 3, & \\ j = 1, 3, & i \neq j \end{pmatrix}. \tag{160}$$

Also in accordance with (155)

$$\theta_{i1} = \theta_{i0} + c_1 t_i + d_1 \qquad (i = 2, 3). \tag{161}$$

From (23) and (18) it follows that

$$\Phi_{ij} = m_i m_j \lambda_i \lambda_j \sum (x_i - x)(x_j - x). \tag{162}$$

Hence equation (156) may be written

$$\begin{aligned}(Cr + E + m_2 \lambda_0)\theta_2 + (Ct + D + m_2 \theta_0)\lambda_2 \\ - m_1 m_2 \lambda_1 \lambda_2 \sum (x_1 - x)(x_2 - x) \\ + (m_1 - m_2) m_2 \lambda_0 \lambda_2 \sum (x_0 - x)(x_2 - x) \\ - m_2 \lambda_2 (t_2 - t)\left(Crt + Dr + Et + F + \frac{m_2 T_0}{m_0} \right) = 0.\end{aligned}$$

We desire to show that the transformation from C_2 to C_{23} satisfies a similar condition, namely

$$\begin{aligned}(Cr_2 + E + m_3\lambda_{20})\theta_{23} &+ (Ct_{23} + D + m_3\theta_{20})\lambda_{23}\\ &- m_1 m_3\lambda_{21}\lambda_{23}\sum(x_{12} - x_2)(x_{23} - x_2)\\ &+ (m_1 - m_3)m_3\lambda_{20}\lambda_{23}\sum(x_{20} - x_2)(x_{23} - x_2)\\ &- m_3\lambda_{23}(t_{23} - t_2)\left(Cr_2 t_2 + Dr_2 + Et_2 + F + m_3\frac{T_{20}}{m_0}\right) = 0.\end{aligned}$$

From equations (16) and (17) we have

$$\lambda_{2i}\theta_2(x_{2i} - x_2) = -(\lambda_i\theta_2 + \theta_{2i}\lambda_2)(x_2 - x) + \lambda_i\theta_2(x_i - x), \tag{163}$$

and similar equations in y, z, and t.

In consequence of the expression for T_{20} analogous to (21), and of equations (16) and (163), the above equation multiplied by θ_2^2 is reducible to

$$\begin{aligned}[C\theta_2 r &+ C\lambda_2(t_2 - t) + E\theta_2 + m_3(\lambda_2\theta_{20} - \lambda_2\theta_0 + \lambda_0\theta_2)]\theta_{23}\theta_2\\ &+ [R(x_2 - x) - (x_3 - x)\lambda_3\theta_2]m_3\{m_1\lambda_1\theta_2(x_1 - x)\\ &+ (m_3 - m_1)\lambda_0\theta_2(x_0 - x)\\ &- (x_2 - x)[m_3(\lambda_2\theta_{20} + \lambda_0\theta_2) + (Cr + E)\theta_2 + C\lambda_2 t_2 + D\lambda_2]\}\\ &+ (Ct_2 + D + m_3\theta_{20})\theta_2(R - \lambda_2\theta_3)\\ &+ m_3[R(t_2 - t) - (t_3 - t)\lambda_3\theta_2]\Big((Ct_2 + D)(\theta_2 r + \lambda_2 t_2 - \lambda_2 t)\\ &+ (Et_2 + F)\theta_2 - \frac{C\theta_2}{m_3} + m_3(\lambda_0\theta_2 + \lambda_2\theta_{20})(t_2 - t)\\ &- m_3\lambda_0\theta_2(t_0 - t)\Big) = 0,\end{aligned}$$

where we have put

$$R = \theta_{23}\lambda_2 + \lambda_3\theta_2.$$

We replace the quantities $\sum(x_i - x)(x_j - x)$, for $i = 1, 2, 3; j = 1, 2, 3; i \neq j$, by their values from (162) and then Φ_{ij} by their values from (160). In like manner we replace $\sum(x_i - x)(x_0 - x)$ for $i = 2, 3$ by their values from (160) and then Φ_{i0} by expressions analogous to (154). Again analogously to [63], [64] we have

$$\sum(x_2 - x)^2 = \frac{2\theta_2}{m_2\lambda_2} + (t_2 - t)^2.$$

When these substitutions are made, we find that the equation is identically satisfied, in consequence of (99), (103), and (121). Hence the relation between C_2 and C_{23} is similar to that of C and C_2. It is evident that the same is true of the transformation from C_3 into C_{23}.

From the generalized theorem of permutability of transformations K [§ 7] it follows that there exists a unique surface S' which is in the relation of transformations K with C_{23}, C_{13}, and C_{12}. However, as yet we do not know that S' is a surface C; but presently this will be seen to be the case.

Now we shall show that θ'_{23}, the function determining the transformation from C_{23} to S', is given by

$$(164) \qquad \theta'_{23} = (\theta_{23})_0 + c_1 t_{23} + d_1,$$

where t_{23} is the complementary function of C_{23}, and $(\theta_{23})_0$, given by an equation analogous to (157), is the function determining a conjugate surface to C_{23}.

From equations analogous to [84] we have

$$\theta'_{23} \lambda_{23} \theta_2 = - \theta_{21} \lambda_2 \theta_3 + \theta_{23} \lambda_2 \theta_1 + \theta_{31} \lambda_3 \theta_2 .$$

Substituting this expression and the expressions for $(\theta_{23})_0$ and t_{23} from equations of the form (157) and (163), we find that in consequence of (161) equation (164) is satisfied identically. Hence for the quatern C_2, C_{12}, C_{23}, S', the conditions of Theorem 11 are satisfied. Consequently S' is a surface C, complementary to C_{23} and we have

THEOREM 12. *If $\bar{S}$ is a special surface Ω and $\bar{S}_1$ and $\bar{S}_2$ are two special surfaces obtained by means of transformations A_{m_1} and A_{m_2} satisfying conditions of the form* (158), *the fourth surface of the quatern, determined by $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$ in accordance with the theorem of permutability of transformations A_m, is a special surface of the same class.*

15. Parallel transformation of special surfaces Ω

Let $\bar{S}$ be a special surface and $\bar{S}_1$ one of its complementary transforms. Also let $\bar{S}_2$ be the parallel transform of $\bar{S}$, as discussed in [§ 10]. In accordance with this theory there exists a fourth surface $\bar{S}_{12}$ such that $\bar{S}_1$ and $\bar{S}_{12}$ are in the relation of a parallel transformation. We desire to show that $\bar{S}_2$ and $\bar{S}_{12}$ are complementary transforms of one another and hence special surfaces Ω.

In analogy with (86) the transformation functions from $\bar{S}$ to $\bar{S}_2$ are given by

$$(165) \quad \theta_2 = - \lambda_2 = 1, \qquad \theta_{20} = \lambda_0, \qquad \lambda_{20} = \theta_0, \qquad \theta_{02} = - \lambda_{02} = 1.$$

The four surfaces C, C_0, C_2, and C_{20}, which are the basis of the transformation from $\bar{S}$ to $\bar{S}_2$ form a quatern such that C and C_2, and C_0 and C_{20} are pairs of associate surfaces. In a similar manner for the quatern C, C_1, C_2, C_{12}, C_1 and C_{12} are associate surfaces, as follows from [§ 10]. Hence we have

$$(166) \qquad \theta_{21} = \lambda_1, \qquad \lambda_{21} = \theta_1, \qquad \theta_{12} = - \lambda_{12} = 1.$$

Now from (38) and an equation analogous to (146) we have

$$(167) \qquad t_2 = r, \qquad r_2 = t.$$

Hence from the foregoing equations

$$(168) \qquad \theta_{21} = \theta_{20} + c_1 t_2 + e_1, \qquad \lambda_{21} = \lambda_{20} + c_1 r_2 + d_1,$$

that is, the surfaces C_2 and C_{12} are complementary transforms of one another, and likewise $\bar{S}_2$ and $\bar{S}_{12}$. However we notice that the constants e_1 and d_1 are interchanged.

By methods analogous to those followed in § 13 it can be shown that the constants F, g, and G are the same for $\bar{S}$ and $\bar{S}_2$. Hence we have

THEOREM 13. *When a special surface Ω whose constants are C, D, E, F, G, g is subjected to the parallel transformation, the resulting surface is a special surface whose constants are C, E, D, F, G, g.*

16. Special isothermic surfaces

We apply the preceding results to the determination and transformation of special isothermic surfaces.*

In this case the lines of curvature being the conjugate system with equal point invariants, we have

$$E = G = e^{2\phi}, \qquad F = 0, \qquad \omega = 45°, \qquad \rho = e^{-2\phi}, \qquad H\rho = 1,$$

ϕ being thus defined. Since t is zero, we take $r = 0$ in conformity with (104), and have from (10) and (4),

$$t = r = 0, \qquad a_0 = b_0 = 0, \qquad w_0 = T_0 = \text{const.} = 2m_0,$$

the constant m_0 being thus determined.

From [114] we have

$$\theta_0 = e^{2\phi}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right), \qquad \lambda_0 = \left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right).$$

As it is desired that C_1 shall be isothermic, it follows from (121) that we must have

$$\frac{c_1}{m_1} = 2 + f_1.$$

If we put

$$A = F, \qquad B = G + \frac{g^2}{4} + Fg + 4C,$$

equation (129) becomes

$$(169) \qquad \Delta_1 \theta_0 + \left(\frac{\lambda_0 \theta_0}{2}\right)^2 + A\lambda_0 \theta_0 - 2D\lambda_0 - 2E\theta_0 + B = 0,$$

* The results which we obtain thus briefly are due to Darboux, *Annales de l'École Normale Supérieure*, ser. 3, vol. 16 (1899), and Bianchi, *Annali di matematica*, ser. 3, vol. 11 (1905), pp. 93–158.

and equation (131) reduces to

$$(170) \qquad m_1(2m_1 + A)^2 - Bm_1 - 2DE = 0.$$

From (5) it follows that if θ_0 and λ_0 are of the form (136) for an isothermic surface, then ρ is a function of v alone and $\bar{S}$ is a surface of revolution, which we have excluded from the discussion. Accordingly an isothermic surface other than a surface of revolution which satisfies the condition (169) is called a *special isothermic surface of class* (A, B, D, E).

From (139) it follows that the transformation functions leading to a complementary transform of $\bar{S}$ are

$$\eta_1 = e^{\phi}\frac{\partial\lambda_0}{\partial u}, \qquad \beta_1 = -e^{\phi}\frac{\partial\lambda_0}{\partial v}, \qquad \mu_1 = w_1 = \tfrac{1}{2}\lambda_0\theta_0 + A + 2m_1,$$

$$\tau_1 = -\theta_1 = -\theta_0 - \frac{D}{m_1}, \qquad \sigma_1 = -\lambda_1 = -\lambda_0 - \frac{E}{m_1}.$$

In order that this value of w_1 shall be equal to that given by (128), we must have $g = 0$.

The results of §§ 11 and 12 when applied to this case enable us to state

THEOREM 14. *A special isothermic surface, whose constants are such that the roots of equation* (170) *are real and distinct, admits three complementary surfaces which are isothermic surfaces of the same class; and the circle-planes of the three transformations coincide.*

From § 13 we have

THEOREM 15. *When $\bar{S}$ is a special isothermic surface, the equations of a transformation D_m of this surface into an isothermic surface admit the first integral*

$$e^{\phi}\left(\eta_1\frac{\partial\lambda_0}{\partial u} - \beta_1\frac{\partial\lambda_0}{\partial v}\right) - (m\lambda_0 + E)\theta_1 - (m\theta_0 + D)\lambda_1$$
$$+ \mu_1(A + 2m + \tfrac{1}{2}\theta_0\lambda_0) = \text{const.}$$

When this constant is zero, the resulting surface is a special isothermic surface of the same class as $\bar{S}$.

From equation (163) for the t's, it follows that if $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$ are isothermic surfaces so also is $\bar{S}_{12}$. Hence

THEOREM 16. *If $\bar{S}_1$ and $\bar{S}_2$ are two special isothermic transforms of $\bar{S}$, there exists a fourth special isothermic surface of the same class which is a transform of $\bar{S}_1$ and $\bar{S}_2$.*

For the present case Theorem 13 becomes

THEOREM 17. *The parallel transform of a special isothermic surface of class (A, B, D, E) is a special surface of class (A, B, E, D).*

17. When θ_0 is linear in t

We consider throughout the remainder of this memoir the case where θ_0 is linear in t with constant coefficients. It follows at once from (5) and (104) that λ_0 is linear in r. Thus we put

$$\theta_0 = c_0 t + d_0, \qquad \lambda_0 = c_0 r + e_0. \tag{171}$$

From (4) we have by integration

$$T_0 = \frac{m_0}{c_0}(\theta_0 \lambda_0 + g_0), \tag{172}$$

where g_0 is a constant of integration.

In consequence of (10) we have from (116) and (128)

$$\begin{aligned} a_1 &= \frac{a_0}{T_0}(c_0 - R_1), \qquad b_1 = \frac{b_0}{T_0}(c_0 - R_1), \\ w_1 &= \frac{w_0}{T_0}(c_0 - R_1) + \frac{m_0(g - g_0)}{w_0}. \end{aligned} \tag{173}$$

It follows from (113) that if g and g_0 in (173) were equal the surfaces C_0 and C_1 would coincide, which evidently is contrary to hypothesis. If we put

$$h = m_0(g - g_0), \tag{174}$$

it follows from the preceding considerations that h does not vary in passing to the various transforms of C.

In considering equations (132) we excluded the case where θ_0 is of the form (171). Referring to (130) and (172), we see that the first two expressions in (132) vanish only when g and g_0 are equal, which we have seen to be impossible. Hence when θ_0 is of the form (171), the vanishing of the second terms of (132) are a consequence of (129) which reduces to

$$\begin{aligned} &h\left\{\rho\left[\left(\frac{\partial t}{\partial u}\right)^2 + \left(\frac{\partial t}{\partial v}\right)^2\right] + 1\right\} \\ &\qquad + 2T_0(Crt + Et + Dr + F + c_0) + jT_0^2 = 0, \end{aligned} \tag{175}$$

where j is the constant given by

$$hj = 2\frac{C}{c_0}(d_0 e_0 + g_0) - 2De_0 - 2Ed_0 + 2Fc_0 + G + c_0^2. \tag{176}$$

If we put

$$\begin{aligned} 2C + jm_0 c_0 &= -hm_0' c_0', \qquad 2D + jm_0 d_0 = -hm_0' d_0', \\ 2E + jm_0 e_0 &= -hm_0' e_0', \\ 2(F + c_0) + j\frac{m_0}{c_0}(e_0 d_0 + g_0) &= -h\frac{m_0'}{c_0'}(e_0' d_0' + g_0'), \end{aligned} \tag{177}$$

equation (175) is reducible to

$$(178)\quad \rho\left[\left(\frac{\partial t}{\partial u}\right)^2+\left(\frac{\partial t}{\partial v}\right)^2\right]+1 = T_0\frac{m_0'}{c_0'}[(c_0' t + d_0')(c_0' r + e_0') + g_0'].$$

Suppose now that we have a surface C satisfying the equations of condition in § 1, with θ_0 and λ_0 of the form (171), and also equation (178). By means of (176) and (177) we can transform the equation (178) to the form (175). Thus from the five arbitrary constants m_0', c_0', d_0', e_0', g_0' we pass to seven, namely C, D, E, F, G, L, and j. Evidently then in all generality h and j can be given fixed values, so long as we take $h \neq 0$. The resulting equations assume a simple form, if we take $j = 0$ and $h = -2$. Now equation (175) becomes

$$(179)\quad \rho\left[\left(\frac{\partial t}{\partial u}\right)^2+\left(\frac{\partial t}{\partial v}\right)^2\right]+1 = T_0(Crt + Et + Dr + F + c_0),$$

and the constants G and g which appear in equation (131) are given by

$$(180)\quad \begin{aligned} G &= 2De_0 + 2Ed_0 - 2Fc_0 - c_0^2 + 2\frac{C}{c_0}(d_0 e_0 + g_0), \\ g &= g_0 - \frac{2}{m_0}. \end{aligned}$$

We recall that every surface C has in general two conjugate surfaces, C_0 and C_0'. From [56] we have that the functions a_0', b_0', w_0' determining C_0' are given by

$$(181)\quad a_0' = \frac{a_0}{w_0^2}, \qquad b_0' = \frac{b_0}{w_0^2}, \qquad w_0' = -\frac{1}{w_0}, \qquad T_0' = \frac{T_0}{w_0^2}.$$

Since the left-hand member of (179) is equal to $H^2\rho^2$, it follows from (10) that this equation may be written

$$(182)\quad \frac{T_0}{w_0^2} = Crt + Et + Dr + F + c_0.$$

In consequence of (106), (181), and (182) equations (173) may be written

$$(183)\quad a_1 = a_0' + \frac{m_1}{m_0}a_0, \qquad b_1 = b_0' + \frac{m_1}{m_0}b_0, \qquad w_1 = w_0' + \frac{m_1}{m_0}w_0.$$

Expressing the condition that these expressions shall satisfy equations (113), when a_0, b_0, w_0 and a_0', b_0', w_0' satisfy it, we get

$$(184)\quad \begin{aligned} m_1\theta_1 &= m_0\theta_0 + m_0'\theta_0', \\ m_1\lambda_1 &= m_0\lambda_0 + m_0'\lambda_0'. \end{aligned}$$

Equations (183) show that each circle-plane of the complementary trans-

formations of C is the plane of the corresponding points M, M_0, M_0' on C, C_0, C_0' respectively, and M_1 lies on the line $M_0 M_0'$.

From equations (184), (171), (99), and (103) it follows that θ_0' and λ_0' are linear functions of t and r respectively.

Conversely, we shall show that when θ_0 and θ_0' are linear functions of t, the surface C is a special surface. If we take

$$\theta_0' = c_0' t + d_0', \qquad \lambda_0' = c_0' r + e_0',$$

the function T_0' is given by an equation analogous to (172), namely

$$T_0' = \frac{m_0'}{c_0}(\theta_0' \lambda_0' + g_0'). \tag{185}$$

From (181) and (10) we have

$$H^2 \rho^2 = T_0 T_0', \tag{186}$$

which is the same as (178) in consequence of (185). Hence we have

THEOREM 18. *When θ_0 and θ_0' for a surface C are linear in t with constant coefficients, C admits three complementary transforms, in general distinct, whose circle-planes are the planes determined by corresponding points on C, C_0, and C_0'; moreover, this is the only type of special surface C with θ_0 linear in t.*

We propose to show that the functions θ_{10} and λ_{10} of a complementary transform are linear in t_1 and r_1 respectively.

We put

$$\theta_{10} = (c_0 + c')t_1 + (d_0 + d'), \qquad \lambda_{10} = (c_0 + c')r_1 + (e_0 + e'). \tag{187}$$

If these equations be multiplied by $m_1 \lambda_1$ and $m_1 \theta_1$ respectively and the expressions for $m_1 \lambda_1 \theta_{10}$, $m_1 \theta_1 t_1$, $m_1 \lambda_1 t_1$, and $m_1 \theta_1 r_1$ from (140), (121), and (146) be substituted, the resulting equations vanish identically if

$$(D + m_1 d_0)c' = (C + m_1 c_0)d',$$

$$\left[\frac{m_1}{c_0}(e_0 d_0 + g_0) + F - c_1\right]c' = (E + m_1 e_0)d' + \left[\tfrac{1}{2}(G + c_0^2) + c_0 c_1 - 2m_1\left(g_0 - \frac{1}{m_0}\right)\right],$$

$$(E + m_1 e_0)c' = (C + m_1 c_0)e',$$

$$\left[\frac{m_1}{c_0}(e_0 d_0 + g_0) + F - c_1\right]c' = (D + m_1 d_0)e' + \left[\tfrac{1}{2}(G + c_0^2) + c_0 c_1 - 2m_1\left(g_0 - \frac{1}{m_0}\right)\right].$$

As these four equations in c', d', e' are consistent, we can find their values such that (187) shall be satisfied. Hence we have

THEOREM 19. *When θ_0 for a special surface C is a linear function of t, the function θ_{10} of each of the complementary transforms of C is linear in t_1.*

From (130), (172), and (180) we have

$$N = c_0 - \frac{2}{T_0}. \tag{188}$$

Hence the transformation functions of complementary transforms, namely (139), are reducible now to

$$\begin{gathered} \mu_1 = c_1 + c_0 - \frac{2}{T_0}, \qquad \tau_1 = -\frac{2t}{T_0} - \frac{d_0 m_1 + D}{m_1}, \\ \eta_1 = -\frac{2\sqrt{\rho}}{T_0}\frac{\partial t}{\partial u}, \qquad \beta_1 = -\frac{2\sqrt{\rho}}{T_0}\frac{\partial t}{\partial v}, \\ m_1 \tau_1 \sigma_1 = \mu_1 (R_1 + c_1) + m_1 \lambda_1 \theta_1. \end{gathered} \tag{189}$$

18. Special surfaces of Guichard

In this section we apply the results of the preceding section to establish the existence of special surfaces of Guichard and to show that they admit transformations.*

When we compare equations (58) and (64), we note that for every surface of Guichard θ_0 and λ_0 are of the form (171) where

$$c_0 = -\frac{1}{2m_0}, \qquad d_0 = e_0 = 0.$$

Again comparing (63) and (172), we see that

$$g_0 = -\frac{1}{4m_0^2}.$$

In consequence of (58), (61), and (63), equations (189) become

$$\begin{gathered} \mu_1 = 2(h - \overline{m}_1), \qquad \tau_1 = -2(e^{\xi} + \tfrac{1}{2}d_1), \\ \eta_1 = -2 \operatorname{csch} \alpha \frac{\partial \xi}{\partial u}, \qquad \beta_1 = -2 \operatorname{sech} \alpha \frac{\partial \xi}{\partial v}, \\ m_1 \tau_1 \sigma_1 = 2D(h^2 - 1)e^{-\xi} + 2e^{\xi} E + 2h(c_1 - F - C) \\ + \left[(\overline{m}_1 + 1)\frac{m_1}{m_0} + 2\overline{m}_1 (F - c_1) + De_1 - 2C \right], \end{gathered} \tag{190}$$

* The results here obtained were established at length by the author in a previous memoir, cf. Annali di matematica, ser. 3, vol. 22 (1914), pp. 191–248.

where $\overline{m}_1$ is the constant given by

$$\overline{m}_1 = 1 - \tfrac{1}{2}\left(\frac{C}{m_1} - \frac{1}{2m_0}\right) = 1 - \tfrac{1}{2}(c_1 + c_0). \tag{191}$$

If the last of (190) be differentiated with respect to u, the resulting equation is reducible by means of (81) to

$$\begin{aligned} m_1 \tau_1 e^{-\xi}(k_1 \phi_1 + \psi_1) + m_1 \sigma_1 e^{\xi} \sinh\alpha + Ee^{\xi}\sinh\alpha \\ + e^{-\xi} D[2h(\cosh\alpha + h\sinh\alpha) + (1 - h^2)\sinh\alpha] \\ + (c_1 - F - C)(\cosh\alpha + h\sinh\alpha) = 0, \end{aligned}$$

where now k_1, as defined by (82), is given by

$$k_1 = \frac{hd_1 + 2e^{\xi}\overline{m}_1}{d_1 + 2e^{\xi_1}}.$$

The consistency of this equation with equations (190) necessitates the relation

$$c_1 - F - C = 4m_1\overline{m}_1, \tag{192}$$

under which condition the last of (190) reduces to

$$\begin{aligned} m_1 \tau_1 \sigma_1 = 2D(h^2 - 1)e^{-\xi} + 2e^{\xi}E + 8hm_1\overline{m}_1 \\ + (De_1 - (1 + \overline{m}_1^2)4m_1). \end{aligned} \tag{193}$$

When these values are substituted in (27), we obtain

$$\begin{aligned} \operatorname{csch}^2\alpha\left(\frac{\partial\xi}{\partial u}\right)^2 + \operatorname{sech}^2\alpha\left(\frac{\partial\xi}{\partial v}\right)^2 \\ + h^2 + D(1 - h^2)e^{-\xi} - Ee^{\xi} + 2Bh + H = 0, \end{aligned} \tag{194}$$

where

$$B = -(2m_1 + 1)\overline{m}_1, \qquad H = 2m_1 + \overline{m}_1^2(1 + 2m_1) - \tfrac{1}{2}De_1. \tag{195}$$

On the other hand equation (179) reduces to (194) where now

$$H = 1 - C - F - c_0 = 2C - 1 + 2\overline{m}_1(2m_1 + 1). \tag{196}$$

Equating these two expressions for H, we have

$$2C + \tfrac{1}{2}De_1 = (1 + 2m_1)(\overline{m}_1 - 1)^2. \tag{197}$$

For the present case we have from (180)

$$G = 2Cc_0 - 2Fc_0 - c_0^2, \qquad g = -c_0^2 + 4c_0,$$

and the cubic (131) becomes

$$2Fc_0 + c_0^2 - 2Cc_0 + 2m_1(4c_0 - c_0^2) + \frac{2CF}{m_1} - \frac{2DE}{m_1} - \frac{C^2}{m_1^2} = 0.$$

In consequence of (191) and (192) this equation is equivalent to (197). With the aid of (195) it can be shown to be equivalent also to

$$(198)\qquad (H - 2m_1)2m_1(2m_1 + 1) - B^2 2m_1 + DE(2m_1 + 1) = 0,$$

which is the equation found in the above-mentioned memoir.

When the above values of the constants are substituted in the expression

$$\tfrac{1}{2}(G + c_0^2) + c_0 c_1 - 2m_1\left(g_0 - \frac{1}{m_0}\right),$$

it vanishes identically. Hence the quantities c', d', e' in (187) are equal to zero, and we have

$$(199)\qquad \theta_{10} = c_0 t_1 + d_0, \qquad \lambda_{10} = c_0 r_1 + e_0.$$

As the functions $\mu_1, \eta_1, \beta_1, \tau_1, \sigma_1$, given by (190) and (193), satisfy equations (81), we have

THEOREM 20. *When a surface of Guichard satisfies the condition* (194), *in which the four constants* B, D, E, *and* H *are such that the roots of equation* (198) *are real and distinct, the surface admits three complementary transforms which also satisfy* (194).

The latter part of this theorem follows from the fact that the transforms are surfaces of Guichard, and from the general results of § 12, since H, as given by (196), depends only on C and F, and B, as follows from (195) and (196), is equal to

$$(200)\qquad B = \tfrac{1}{2}(3C + F + c_0 - 2).$$

When a surface of Guichard satisfies (194) we say that it is a *special surface of class* (B, D, E, F).

Since $c_0 = -\frac{1}{2}m_0$ and $d_0 = 0$, it follows from (64) that

$$(201)\qquad r = -(1 + h)e^{-\xi}.$$

When $\bar{S}$ is a special surface of Guichard and $\bar{S}_2$ is a surface of Guichard which is a transform of $\bar{S}$ in accordance with §§ 5 and 6, then in consequence of the formulas in these sections and the present one, equation (158) is reducible to

$$(202)\qquad \begin{aligned} &\operatorname{csch}\alpha\frac{\partial\xi}{\partial u}\eta_2 + \operatorname{sech}\alpha\frac{\partial\xi}{\partial v}\beta_2 - \left(\frac{D}{2} + m_2 e^{\xi}\right)\left(\sigma_2 + \frac{\mu_2^2}{\tau_2}\right) \\ &\qquad + \tau_2\left[m_2(1 - h^2)e^{-\xi} + \frac{E}{2}\right] - \mu_2[B + h(1 + 2m_2)] = 0. \end{aligned}$$

Hence for all transformations from a special surface of Guichard into a surface of Guichard the left-hand member of this equation is constant. Applying Theorem 10 to this case, we have

THEOREM 21. *When a special surface of Guichard undergoes a transformation A_m for which the functions satisfy* (202), *the resulting surface is a special surface of Guichard of the same class.*

If $\bar{S}$, $\bar{S}_1$, and $\bar{S}_2$ are three surfaces of Guichard of which $\bar{S}_1$ and $\bar{S}_2$ are transforms of $\bar{S}$, the fourth surface of the quatern, determined in accordance with the general theorem of permutability of transformations A_m is a surface of Guichard. We shall not go through the details of establishing this here.* Hence in view of Theorem 12 we have

THEOREM 22. *If $\bar{S}_1$ and $\bar{S}_2$ are two transforms of a special surface of Guichard $\bar{S}$ satisfying* (202), *the fourth surface of the quatern is a special surface of the same class.*

Combining Theorems 3 and 13, we have

THEOREM 23. *The parallel transform of a special surface of Guichard of class (B, D, E, H) is a special surface of class (B, E, D, H).*

19. ENVELOPE OF THE CIRCLE-PLANES OF A COMPLEMENTARY TRANSFORMATION

We consider now the envelope of the circle-planes of a complementary transformation of a special surface.

With the aid of (34), (41), and (139) the expressions for the functions p, ω, and ψ, as given by (47), (50′), and (51), are reducible to

$$(203)\qquad \begin{gathered} p = \frac{\theta_0}{\nu}, \qquad \omega = \left[\theta_0(Dr + F - c_1) - (Ct + D)\left(\frac{T_0}{m_0} - \lambda_0 t\right)\right]\frac{1}{\nu}, \\ \psi = \frac{\theta_1}{\nu}\left[C\left(\lambda_0 t - \frac{T_0}{m_0}\right) + \lambda_0 D - m_1 g\right], \\ \nu = \theta_0(Cr + E) - \lambda_0(Ct + D). \end{gathered}$$

We recall that the linear element of the envelope S_0 is

$$ds_0^2 = d\omega^2 + 2dpd\psi.$$

Hence, if we put

$$(204)\qquad \omega = x, \qquad p = -y + iz, \qquad 2\psi = -(y + iz),$$

the surface whose coördinates x, y, z, are given by (204) is applicable to S_0.

When in particular $\bar{S}$ is a special isothermic surface, the expressions (203) are reducible in consequence of the results to § 16, to

$$p = \frac{\theta_0}{\nu}, \qquad \omega = -\frac{2}{\nu}(m_1\theta_0 + D),$$

$$\psi = \frac{m_1\theta_0 + D}{\nu}(\lambda_0 d_1 - 2c_1), \qquad \nu = \theta_0 E - \lambda_0 D.$$

---

* Cf. Annali, l. c., p. 213.

Substituting these values in (204) and eliminating θ_0 and λ_0, we get

$$(y + iz)[x - 2m_1(y - iz)] = 2x[(2C + d_1 E)(y - iz) - c_1 x + d_1].$$

Hence S_0 is applicable to a quadric, as Darboux has shown.*

When $\bar{S}$ is a special surface of Guichard, in consequence of the results of § 18, the expressions (203) are reducible to

$$p = \frac{e^\xi}{2\nu}, \qquad \omega = -\frac{2m_1\bar{m}_1 e^\xi + Dh}{\nu},$$

$$\psi = -\left[m_1(1 - m_1)e^\xi + \frac{D}{2}(1 - h)\right]\frac{d_1 e^{-\xi}(1 + h) + 2(1 + \bar{m}_1)}{\nu},$$

$$\nu = \tfrac{1}{2} Ee^\xi + \tfrac{1}{2} De^{-\xi}(1 - h^2).$$

These are equivalent to the expressions for p, ω and ψ which we have found previously, and from which we showed that S_0 is applicable to a quadric.†

20. The Case $m_1 = m_2$

In this closing section we consider the case where two transformations A_m, with the same values of the constant m, have the same circle-planes. From (93) and (98) it follows that in this case θ_0 and λ_0 are of the form (171).

If we put $m_1 = m_2 = m$, equations (98) may be replaced by

$$\theta_2 = \theta_1 + \frac{\kappa m_0}{m}\theta_0, \qquad \lambda_2 = \lambda_1 + \frac{\kappa m_0}{m}\lambda_0. \tag{205}$$

From these equations, (95) and (10) we have the first two of the equations

$$a_2 = a_1 + \kappa a_0, \qquad b_2 = b_1 + \kappa b_0, \qquad w_2 = w_1 + \kappa w_0, \tag{206}$$

the last being a consequence of the fact that a_2 and b_2 as thus given must satisfy equations (113) for $i = 2$, when a_1 and b_1 satisfy these for $i = 1$.

When we express the condition that θ_2 and θ_1 shall satisfy equations of the form (36), the resulting equations are reducible in consequence of (10) and (171) to the single one

$$m\lambda_2 t_2 = m\lambda_1 t_1 - \kappa m_0 d_0 r + \kappa B_0, \tag{207}$$

where

$$B_0 = \frac{c_0}{m} - \frac{g_0}{c_0} - \frac{e_0 d_0}{c_0}.$$

It is readily shown that this value of t_2 satisfies equations (38) for $i = 2$ on the assumption that t_1 satisfies them for $i = 1$.

* L. c.; cf. also Bianchi, l. c., p. 139.

† Annali, l. c., p. 241.

The remaining condition to be satisfied in order that the transform of C by a_2, b_2, ω_2, θ_2, λ_2 shall be a surface C is equation (115) for $i = 2$. This necessitates the relation

$$
\begin{aligned}
(208) \qquad m_0(\lambda_0\theta_1 + \theta_0\lambda_1) - \Phi_{10} - \left(T_0 - \frac{m_0 c_0}{m}\right) m\lambda_1(t_1 - t) \\
= \kappa\frac{m_0^2}{m}\left(g_0 - \frac{c_0^2}{2m}\right).
\end{aligned}
$$

If the left-hand member of this equation be differentiated with respect to u and v separately, the resulting equations vanish identically in consequence of (36) and (96). Hence for any transformation K_m of a surface C satisfying (171) the left-hand member of (208) is a constant. If this constant is different from zero, a constant κ can be found so that (208) holds, and then (205) and (206) define a second transformation such that for the two transformations the circle-planes coincide. Accordingly we have

THEOREM 24. *When the function θ_0 of a surface S is linear in t, each transformation A_m for whose functions the left-hand member of* (208) *is different from zero has associated with it another transformation A_m such that the circle-planes of the two transformations coincide.*

The surfaces of Guichard are of this type.

PRINCETON UNIVERSITY,
May 14, 1915.